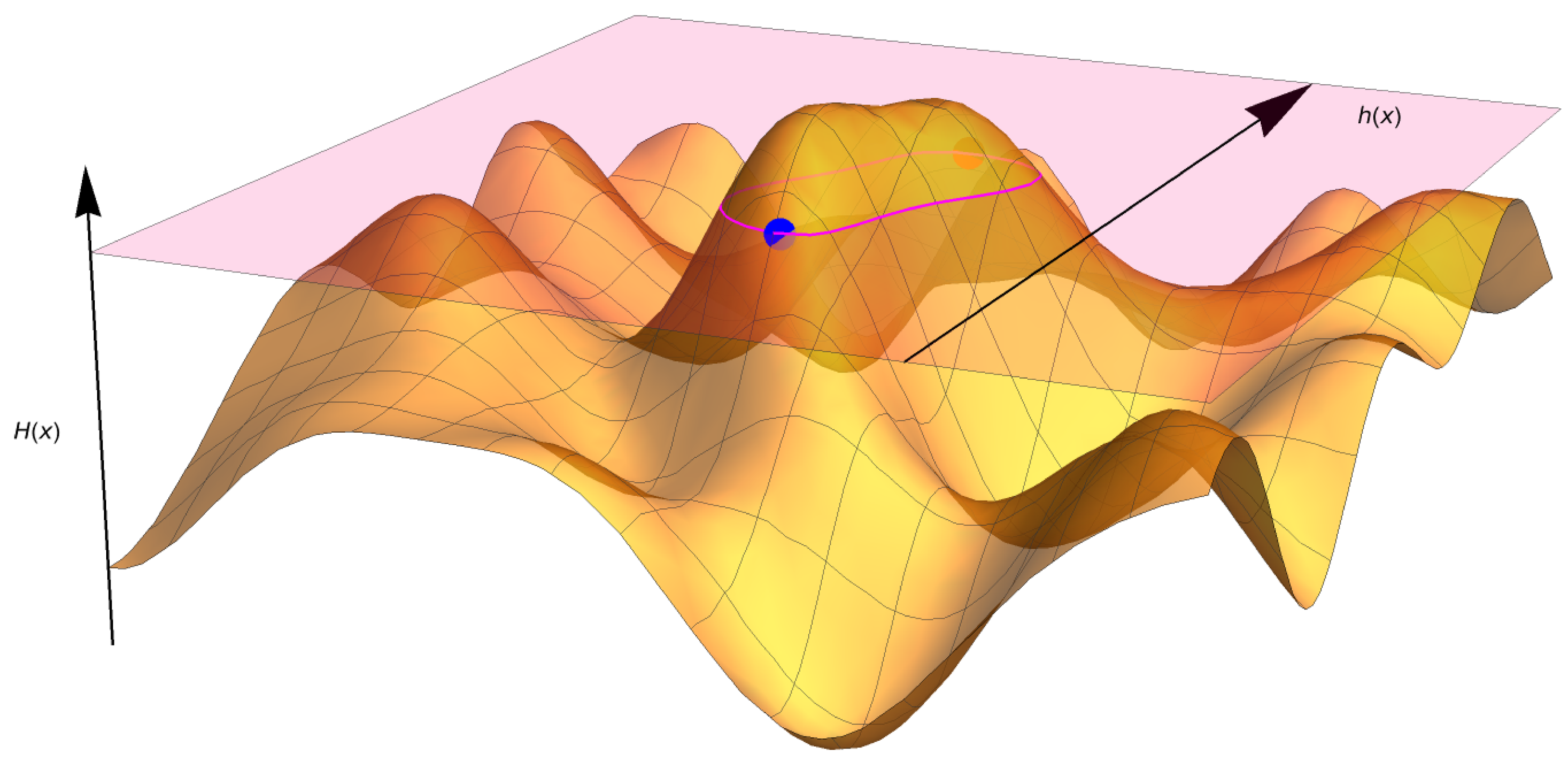
New characterizations of level-set topology in mean-field complex landscapes

Jaron Kent-Dobias • ICTP SAIFR & IFT Unesp

Structure in random landscapes has long been quantified using the statistics of stationary points. While this analysis gives some information about the propensity of a system to glassy dynamics, it doesn't lead to definitive conclusions. Moreover, many interesting problems in inference and machine learning have landscapes with large flat bottoms, and one cannot speak of stationary *points*. Better understanding landscape properties requires new tools for measuring their structure. We introduce three new ways to quantify the properties of level sets in landscapes, including the zero-cost solution sets of inference problems.

Euler characteristic of level sets

On the topology of solutions to random continuous constraint satisfaction problems, JK-D, SciPost Physics **18**, 158 (2025)



Consider a function $H : \mathbb{R}^N \to \mathbb{R}$. The level set of H at energy density E is $\Omega = \{x \in \mathbb{R}^N \mid H(x) = NE\}$. The Euler characteristic $\chi(\Omega)$ of the level set is a topological invariant given by the alternating sum over its Betti numbers. Morse theory tells us that for almost all functions $h : \Omega \to \mathbb{R}$, we can compute the Euler characteristic by the alternating sum of the number of stationary points of increasing index. Therefore

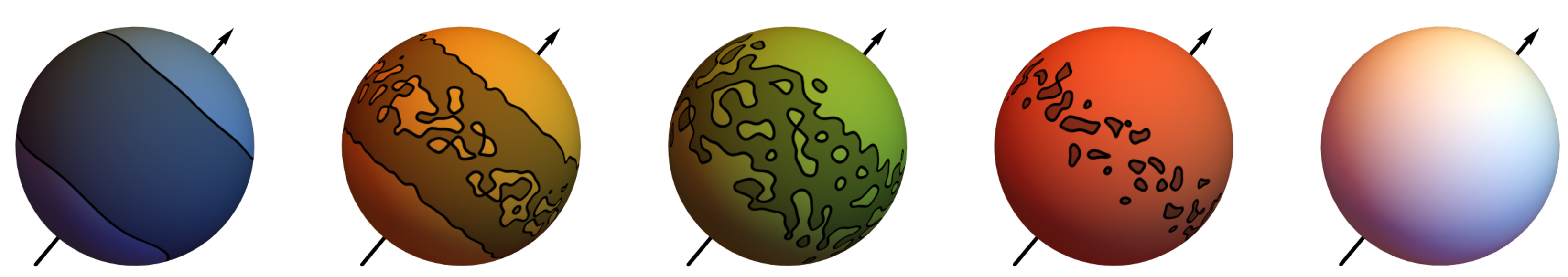
$$\chi(\Omega) = \sum_{\text{ind}=0}^{N} (-1)^{\text{ind}} \mathcal{N}_{\text{ind}}[h] = \int_\Omega dx\, \delta(\nabla h(x)) \det \operatorname{Hess} h(x)$$

A Langrange multiplier can be used to treat the integral over Ω. Defining the Langrangian $L(x, \omega) = h(x) + \omega(H(x) - NE)$, we have

$$\chi(\Omega) = \int_{\mathbb{R}^N} dx \int_{\mathbb{R}} d\omega\, \delta(\partial L(x, \omega)) \det \partial\partial L(x, \omega)$$

where $\partial = [\frac{\partial}{\partial x}, \frac{\partial}{\partial \omega}]$ is the derivative in the space of configurations x and multipliers ω. This can be calculated for mean-field random functions H.

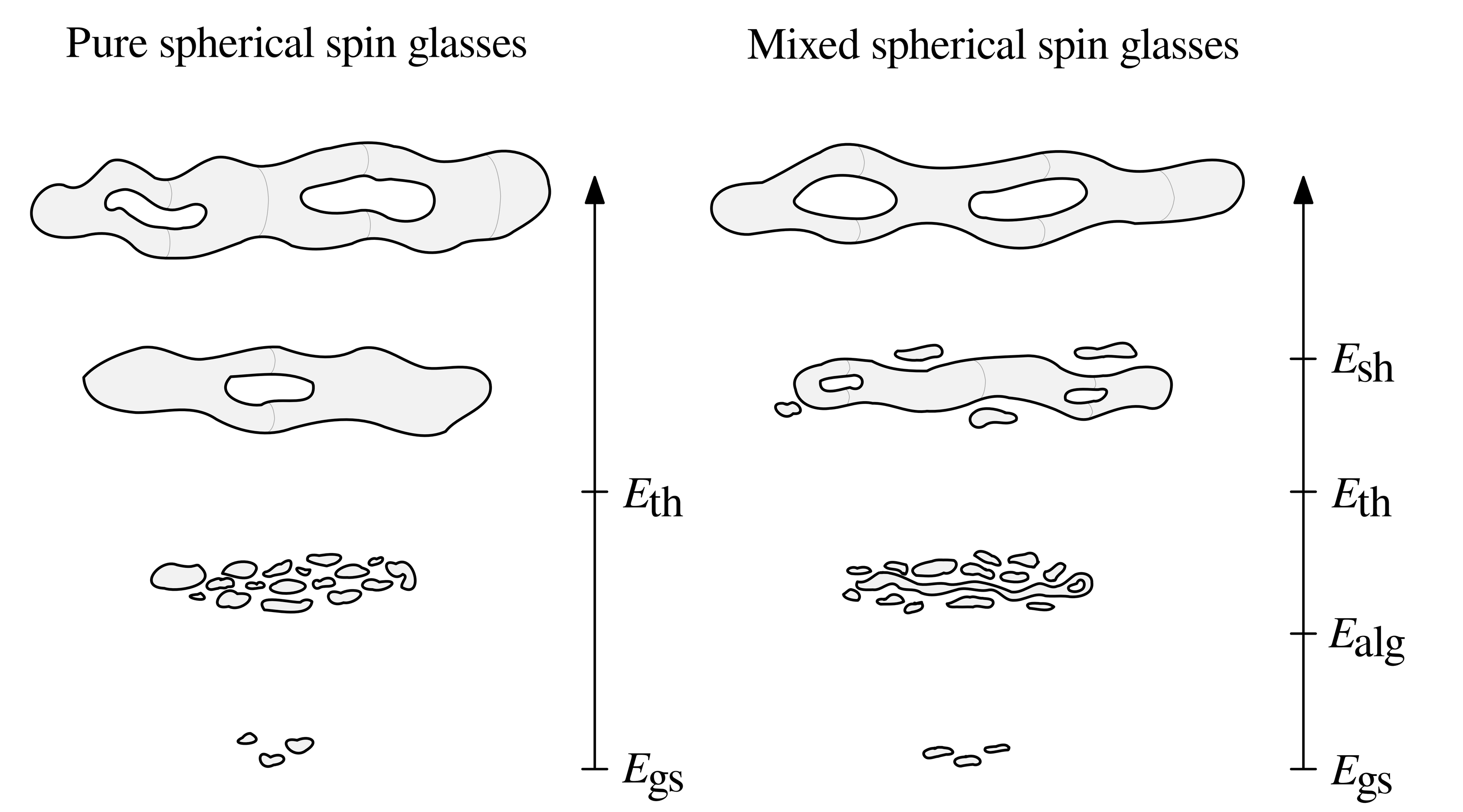
In complex landscapes χ is typically exponentially large in N, and can be shown to transition from negative to positive at an energy density E_{sh}. Perhaps surprisingly, E_{sh} is different from E_{th} (the energy density where most stationary points transition from being minima to being saddles), and appears to usually be larger.



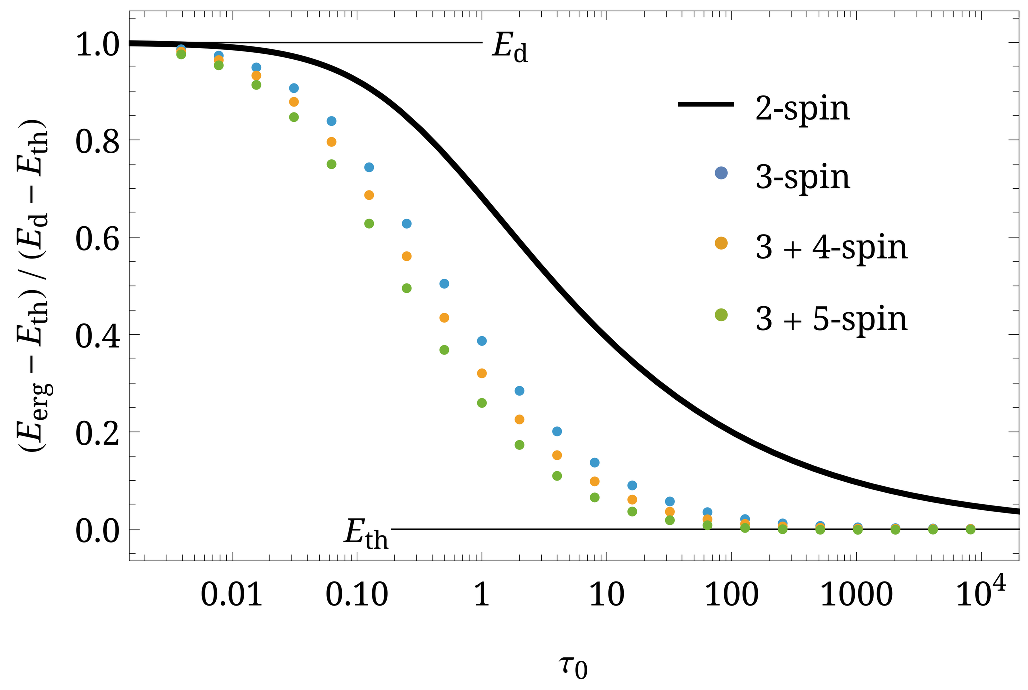
This method works equally well to describe the solution set of continuous constraint satisfaction problems with equality constraints, like $H^\mu(x) = E^\mu$ for $\mu = 1, \ldots, M$, typical of zero-cost solutions to neural networks with smooth activations. Simply employ M Lagrange multipliers in the expressions above and everything follows. Several regimes can be inferred from this.

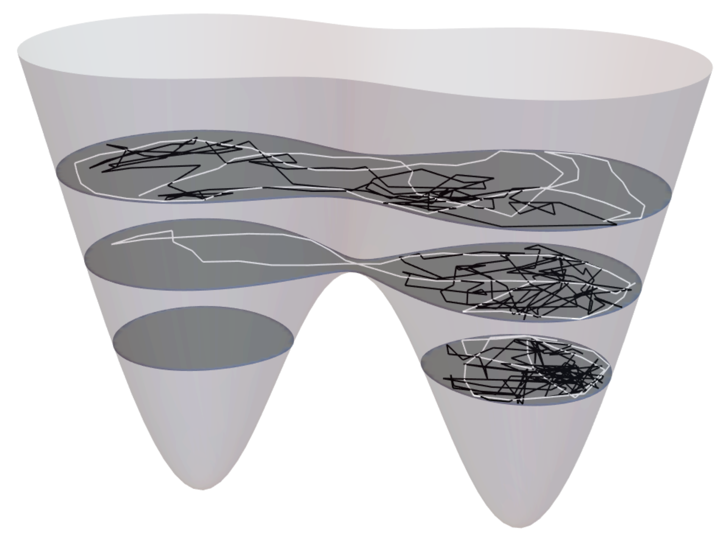
Overlap gap for typical components of the level set

Very persistent random walkers reveal transitions in landscape topology, JK-D, Physical Review Letters **136** 117401 (2026)



The overlap gap property posits that the lack of *any* system-spanning component of the level set impedes optimization by most methods, and sophisticated algorithms can saturate the bound E_{alg} it implies. However, most algorithms used in practice are not sophisticated: they look like gradient descent, perhaps the worst-performing algorithm that terminates in a minimum. We conjecture that gradient descent is likely to stop at the lowest energy at which *typical* configurations belong to a system-spanning component.

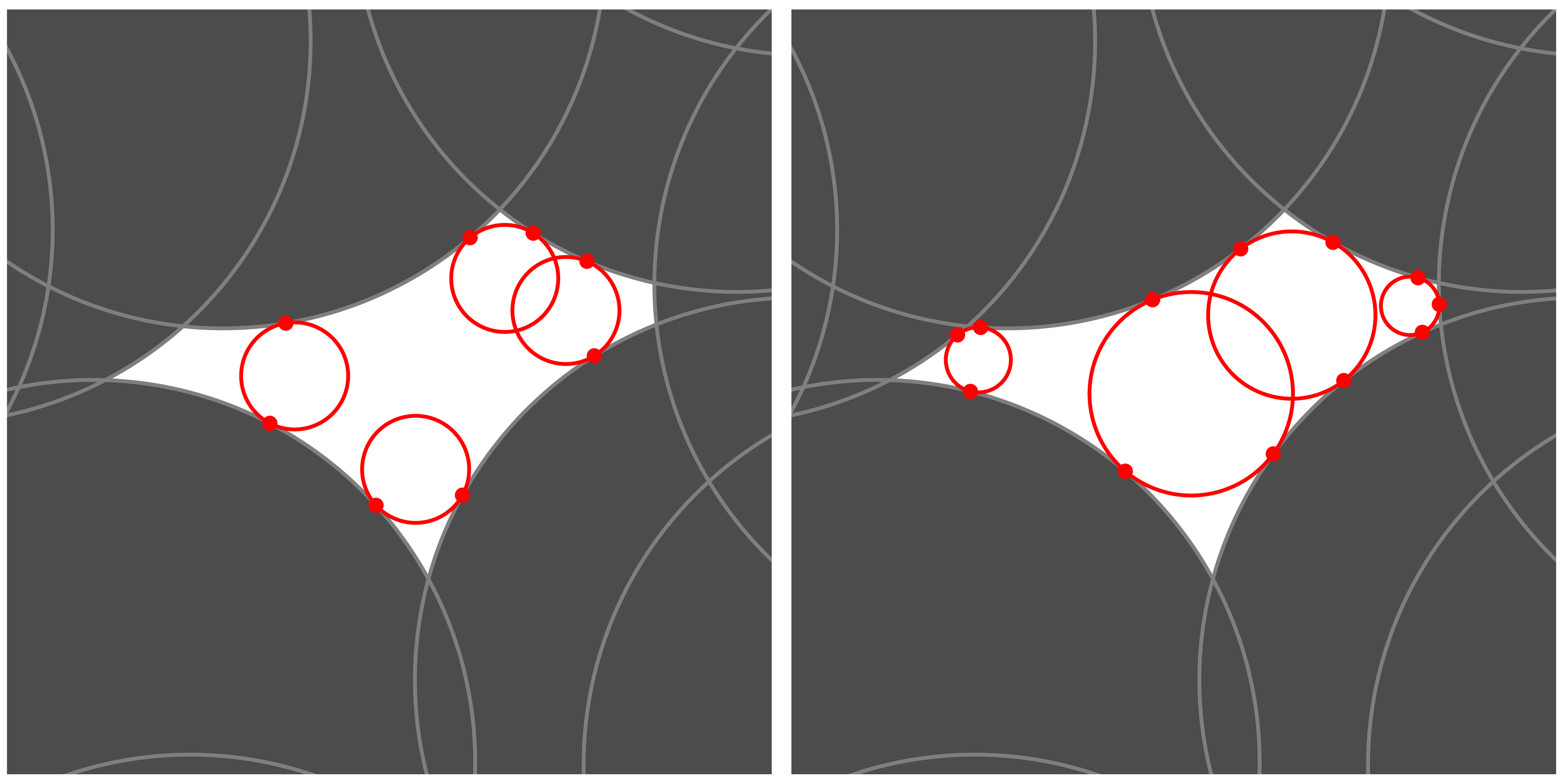




Such connectivity property can be probed by the behavior of random walks confined to the level set and starting from a typical configuration. We develop a DMFT for such walks in mean-field systems, and show that the connectivity properties are determined by the ergodicity of active random walks in the limit of infinite persistence τ_0. Using this tool, we show that for mixed $p + s$ spherical spin glasses with $p \simeq s$, belonging to a spanning component becomes atypical at E_{th}. The situation must change for $p \ll s$ since $E_{\text{alg}} > E_{\text{th}}$, a topic of ongoing research.

Statistics of wedged and inscribed spheres

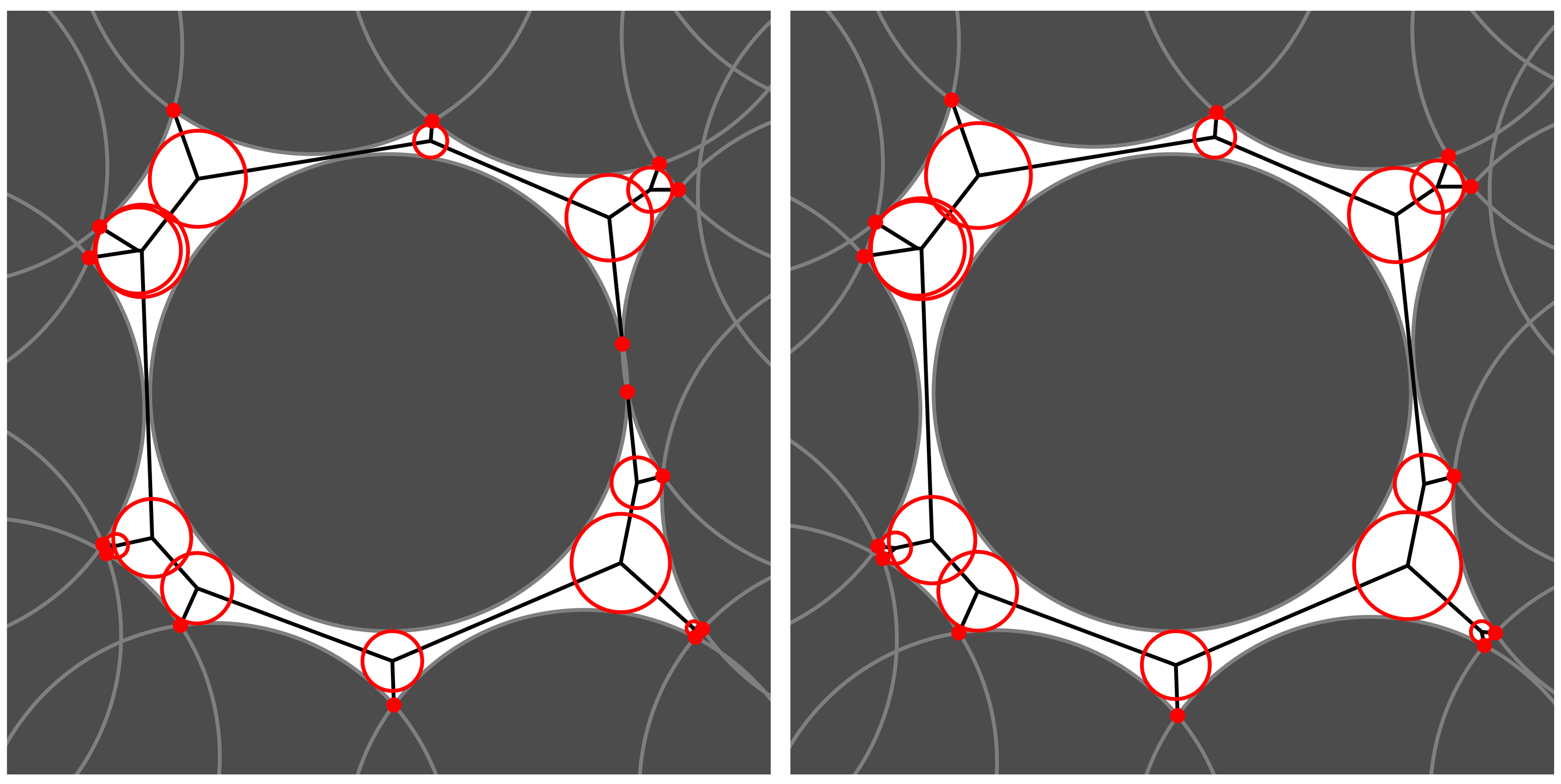
Structure of solutions to continuous constraint satisfaction problems through the statistics of wedged and inscribed spheres, JK-D, Journal of Statistical Mechanics: Theory and Experiment **2026** 023301 (2026)



In constraint satisfaction problems with inequality constraints like $H^\mu(x) \geq E^\mu$, the solutions set is not a manifold but instead has a non-smooth boundary. We can study its properties by the statistics of spheres wedged and inscribed into the solution set. Like stationary points, these can be counted and their clustering properties characterized.

The relative counts of wedged and inscribed spheres constrain the topology of the solution space. The set of such spheres forms a graph which, assuming the jamming transition is isostatic, only contains internal vertices of degree $N+1$ and leaves. The graph is a tree if the solution space is N-simply connected, and then the ratio of counts is of order N. When the solution space is not N-simply connected, the graph can gain loops, which leads to a deficit of wedged spheres compared to inscribed spheres. When the ratio of counts is very different from order N, a topologically nontrivial phase is implied.

We calculate these counts for the negative margin spherical perceptron, where we show when the solution set is loopy or not.



JK-D is supported by FAPESP Young Investigator Grant No. 2024/11114-1 and by the Simons Foundation Targeted Grant to ICTP-SAIFR.