How to count in hierarchical landscapes

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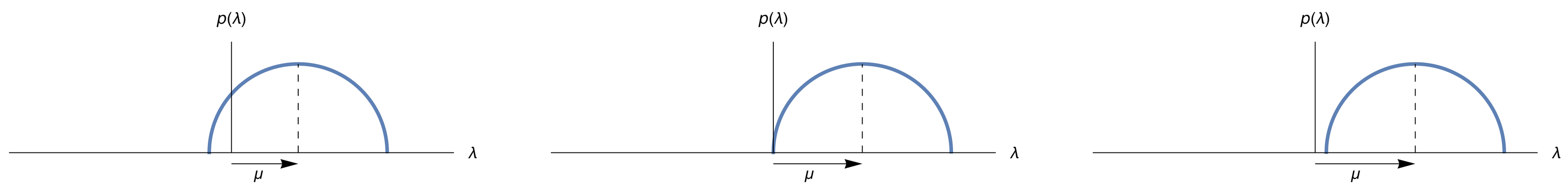
The energy landscape and its geometry are thought to influence glassy behavior via the proliferation of metastable states. However, the distribution and properties of these states is only understood in the simplest mean-field models. We show how to count states with different properties in slightly more complicated mean-field models. We also share preliminary results on the relative arrangement of nearby states.

Counting different types of states

The *mixed spherical models* are a family of mean-field glass models containing all isotropic Gaussian functions on the N-dimensional hypersphere. Inside this family, virtually any hierarchical structure (type of RSB) can be found. Each member model is defined by the covariance between the energy at two points

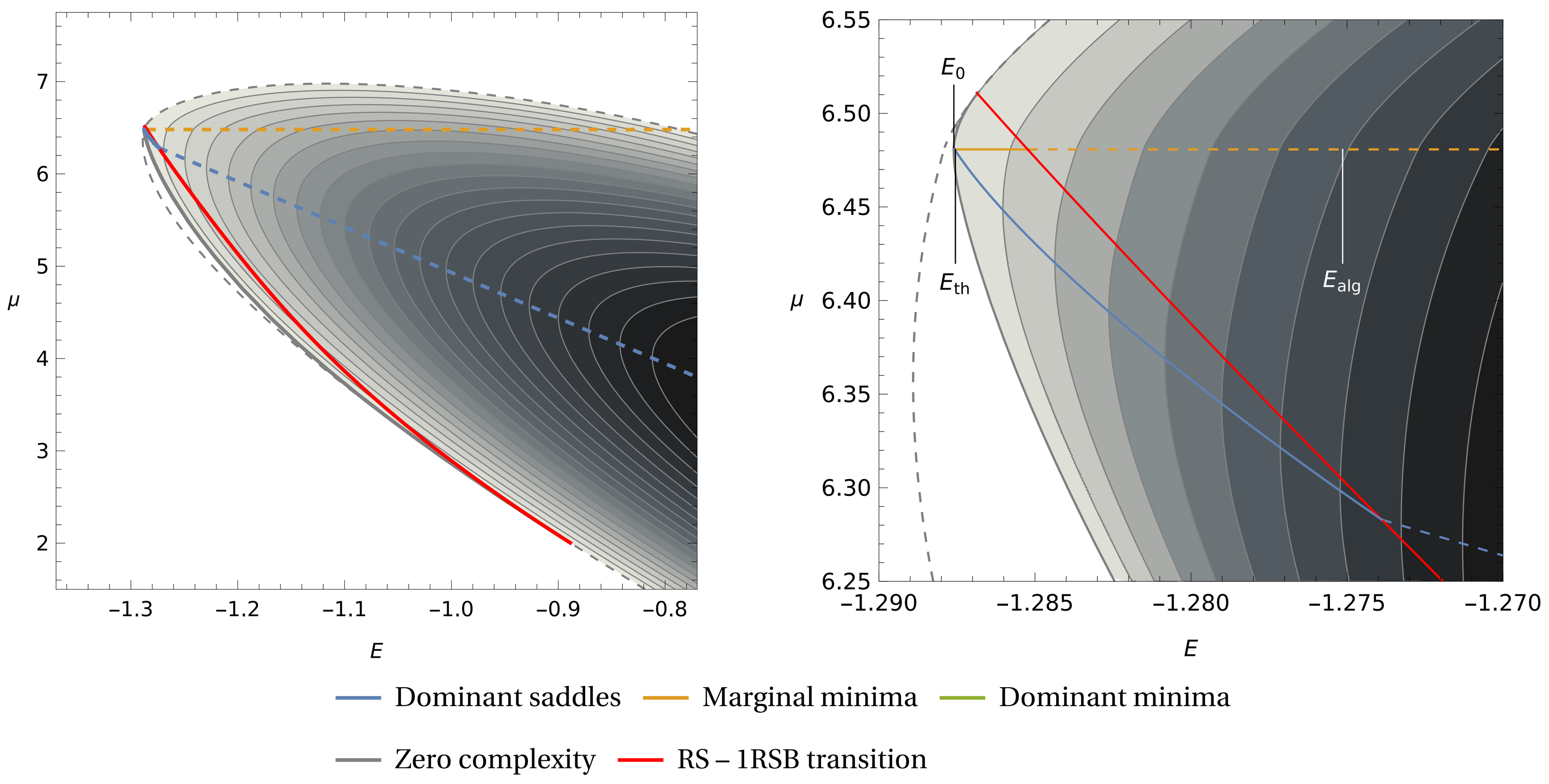
$$\overline{H(\mathbf{s})\cdot H(\boldsymbol{\sigma})}=\frac{1}{N}f\left(\frac{\mathbf{s}\cdot\boldsymbol{\sigma}}{N}\right)$$

We take $f(q)=\frac{1}{2}(a_pq^p+a_sq^s)$ and call the result the "$q+s$ model."



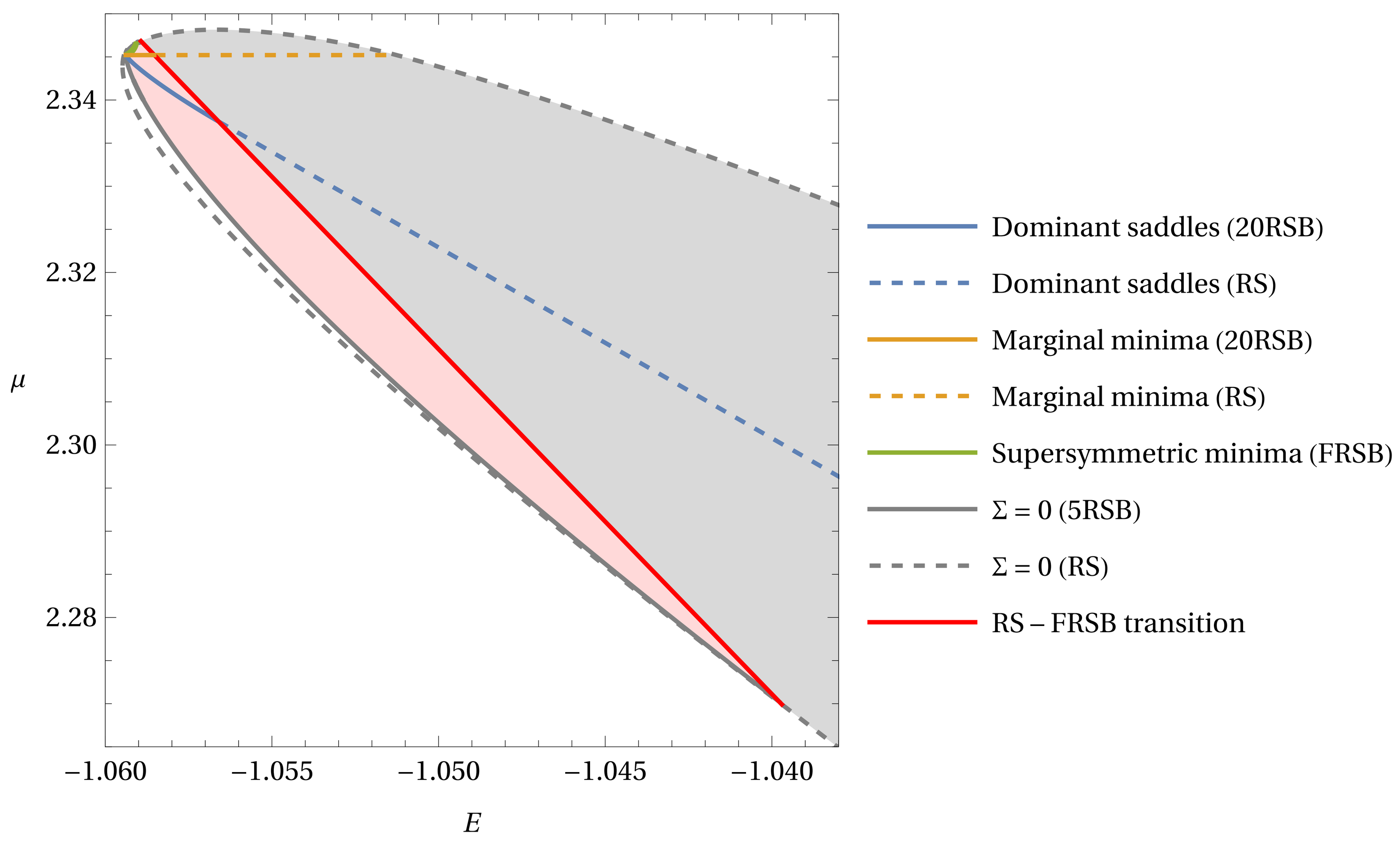
We count *stationary points*, where $\nabla H=0$, including minima, maxima, and saddle points. In these models, stationary points have two important properties: their *energy* $E=H/N$ and their *index* (the number of unstable directions). We use a proxy for index: the *stability* μ, which fixes the eigenvalue spectrum of the Hessian. The spectrum touches zero at the marginal stability μ_m. When $\mu<\mu_\text{m}$, the stationary point is a saddle with many downward directions, and when $\mu>\mu_\text{m}$ it is a minimum.

We show how to count stationary points in two models with nontrivial hierarchy of states: a $3+16$ model with 2RSB in equilibrium, and a $2+4$ model with full RSB in equilibrium. Generically, finite kRSB in equilibrium corresponds with $(k-1)$RSB of the ground state structure, so we expect the structure of the metastable states to usually be at most $(k-1)$RSB.



This plot shows the *complexity* (the logarithm of the number of states) with given properties for the $3+16$ model. Dashed lines show the annealed solution, which is sometimes not correct, and solid lines show the 1RSB solution. An RSB phase begins at high energy densities among high-index saddles, eventually extending to a small subset of minima above the marginal stability.

In this model, the *threshold energy* E_th, where most stationary points are marginal, is significantly below the algorithmic limit E_alg, where no smooth dynamics can pass. This is a conclusive example of a situation where the threshold, once thought to be significant for long-time dynamics, is actually irrelevant to the dynamics.

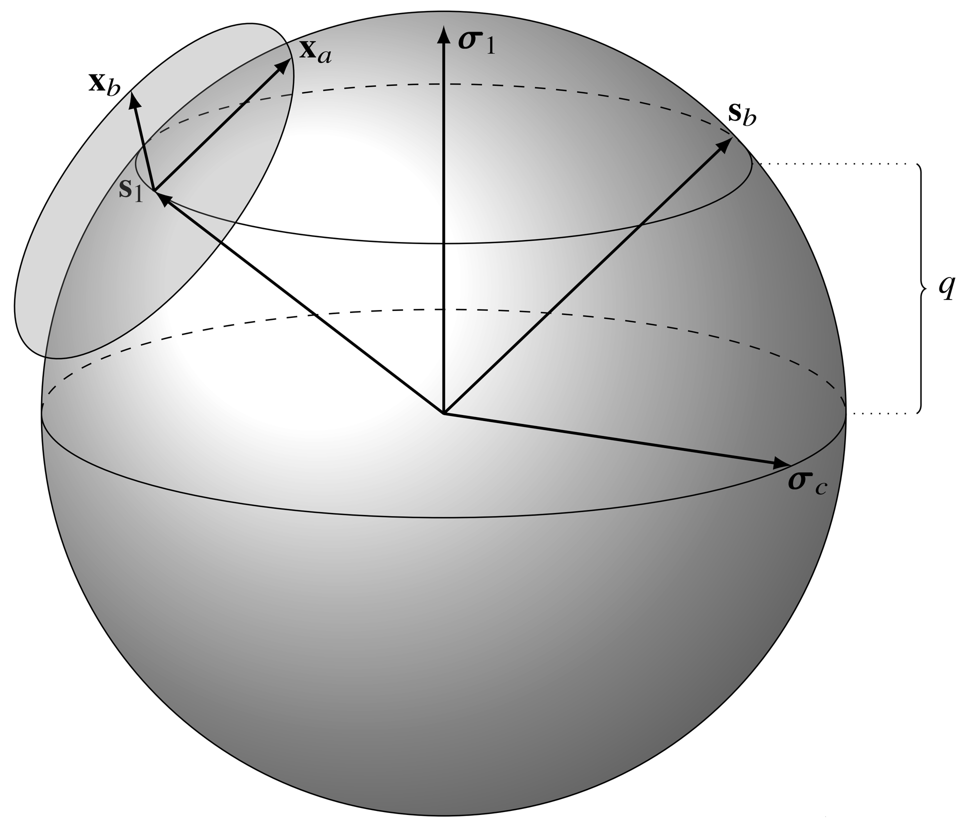


This plot shows the phases of structure in the $2+4$ model, where the red region now denotes a full RSB solution. This solution correctly predicts the location of the ground state, which is marginal as expected. The lines on this plot were produced with a 20RSB approximation.

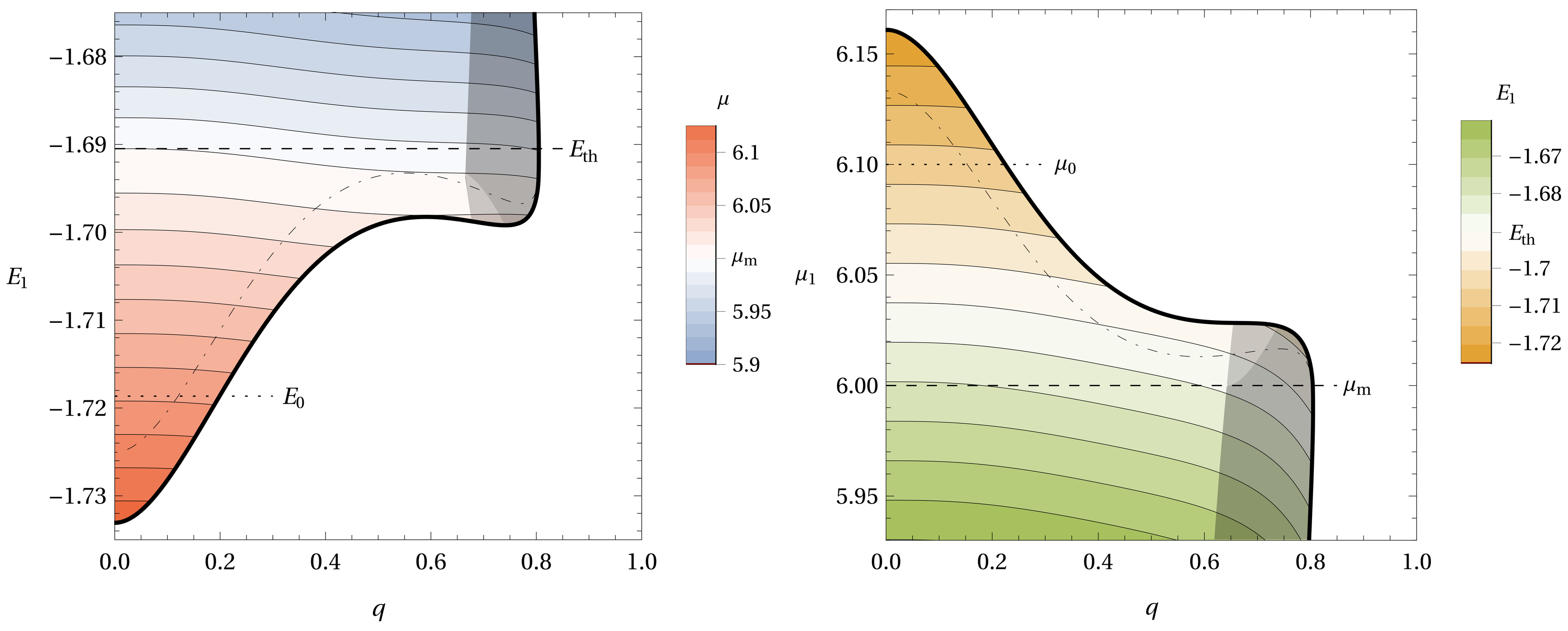
Relative arrangement of the states

On the left, we counted states. But more geometric information is possible! Given a state $\boldsymbol{\sigma}$ with certain properties, how many states of other properties lie a fixed distance away? Instead of distance we use the *overlap* q, which is the normalized dot product of the two state vectors.

This has been computed for the simplest models, and we extend this work to the simplest of the mixed models. Here, we focus on a model with $f(q)=\frac{1}{2}(q^3+q^4)$, with a 1RSB equilibrium and a replica-symmetric complexity.

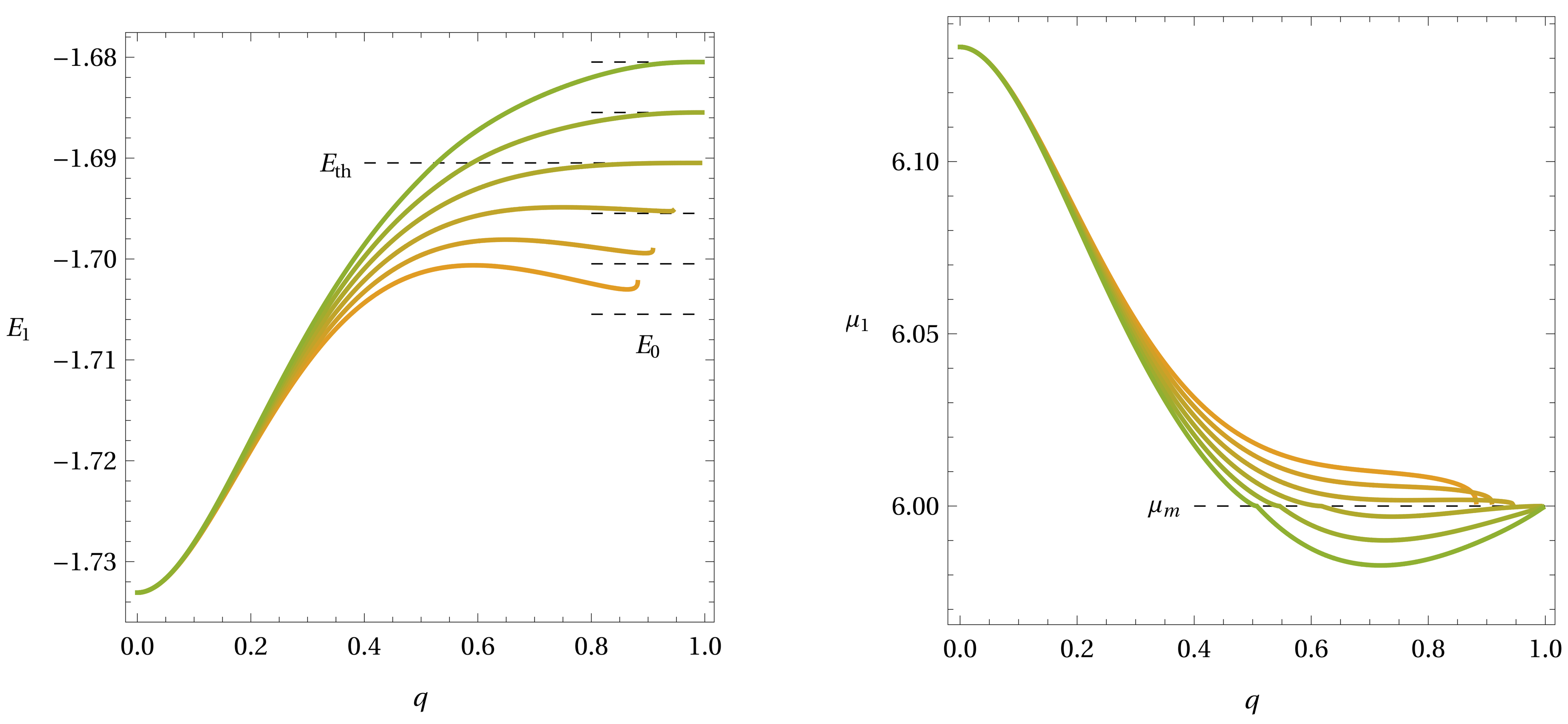


Requiring that states lie nearby each other can qualitatively change their spectrum. Since the condition of proximity produces only a rank-one modification of the Hessian matrix, the continuous part of the spectrum is not affected. However, an isolated eigenvalue can be forced from the bulk by the conditioning, potentially destabilizing what would otherwise look like a minimum. When this happens, the associated eigenvector is typically correlated with the direction between the two nearby states.



These plots show the neighbors of a stable reference minimum. The left plot shows the most common states with given energy and overlap, while the left shows the most common states with given stability and overlap. The shaded regions show the behavior of an isolated eigenvalue. The lightly shaded regions correspond to minima with an isolated eigenvalue that does not change their stability. The darkly shaded regions correspond to saddles with an isolated eigenvalue, and either have many other unstable directions or are naïve minima destabilized by the isolated eigenvalue.

Of special interest in the mixed models are the marginal states with $\mu=\mu_\text{m}$. Evidence suggests that long-time quench dynamics arrives at marginal states, but we don't know how to predict which ones. As noted, those at the threshold energy were thought to be important, but recently have been shown to not attract the dynamics in most models.



There are significant geometric differences among the marginal states, but these changes all revolve around the debunked threshold energy. This plot shows the minimal-energy neighbors of marginal states with various E_0 as a function of overlap. For energies above the threshold, there are states at arbitrarily small distance, and they have $\mu<\mu_\text{m}$. Below the threshold, a gap appears, and the nearest states have $\mu>\mu_\text{m}$.