NOVEL CRITICAL PHENOMENA

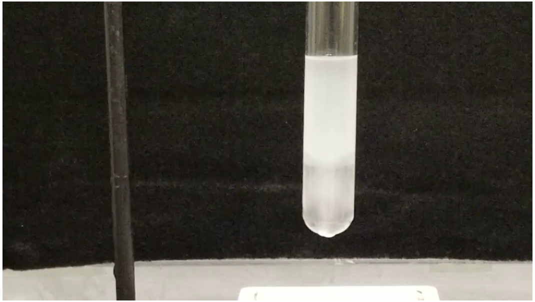
Jaron Kent-Dobias

A dissertation presented to the faculty of
the Graduate School of Cornell University

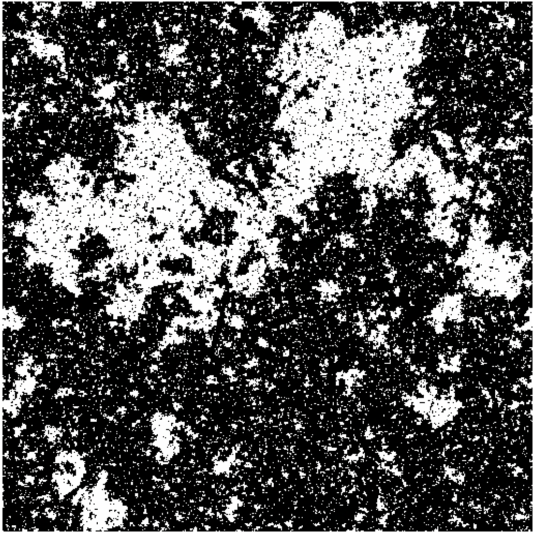
15 June 2020

Introduction

What are critical phenomena?



Critical Opalescence 2019 (no audio), Chemistry Demo Lab Ohio State University (2019)



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A canonical example: percolation

Percolation studies the connectivity of randomly depleted networks.

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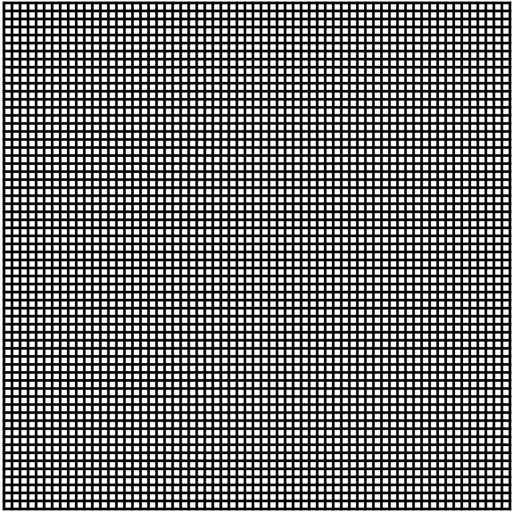

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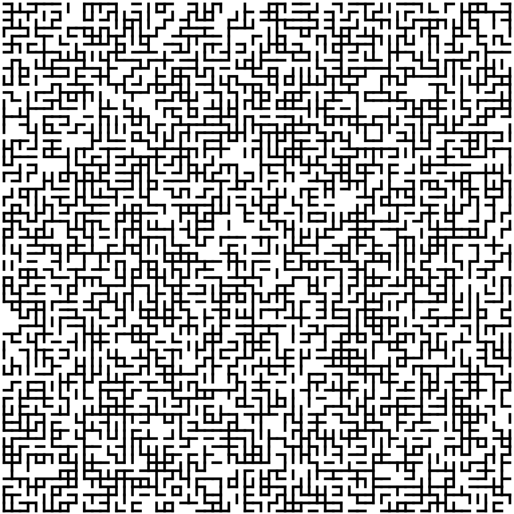

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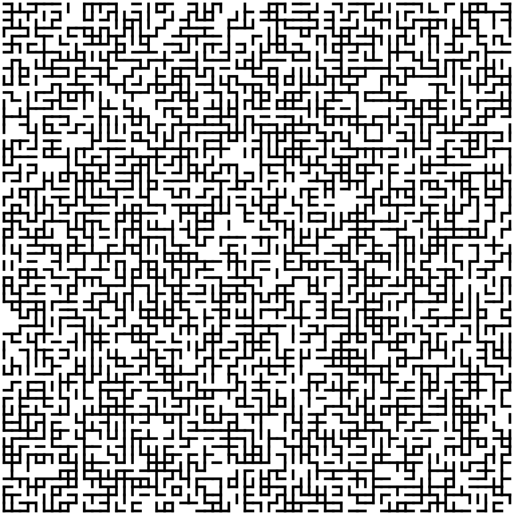

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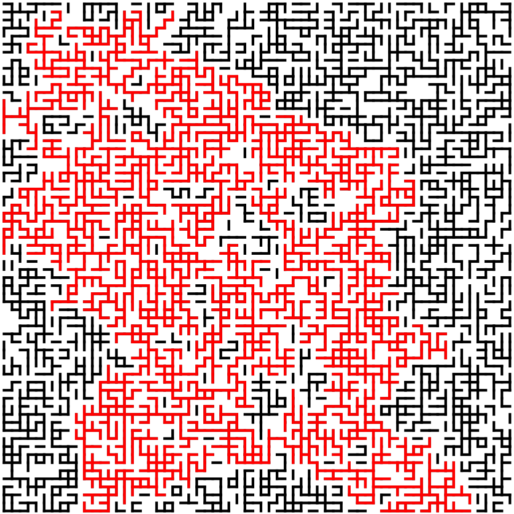

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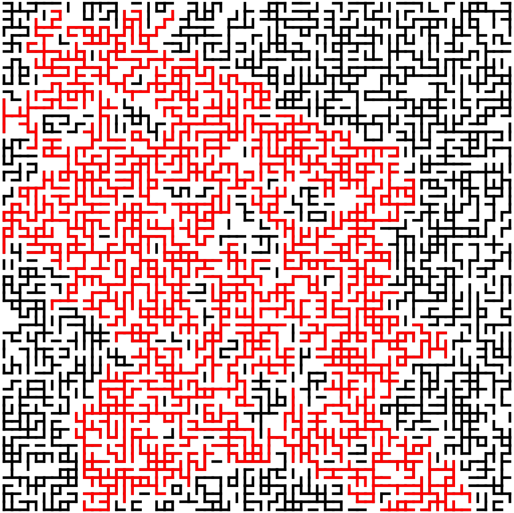
Continuous transition from connected to disconnected at the critical point $p = p_c$.

- For $p < p_c$, clusters of bonds have a typical maximum size.
- For $p > p_c$, non-spanning clusters have a typical maximum size.
- At the critical point, clusters have *no* typical size.



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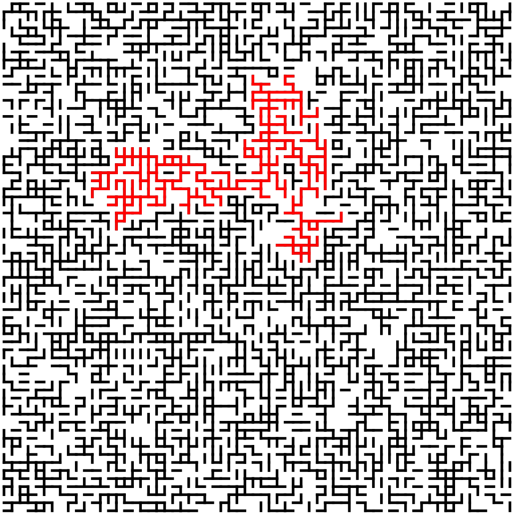

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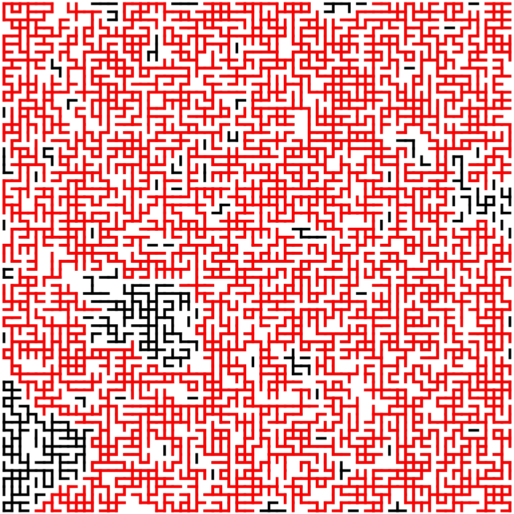

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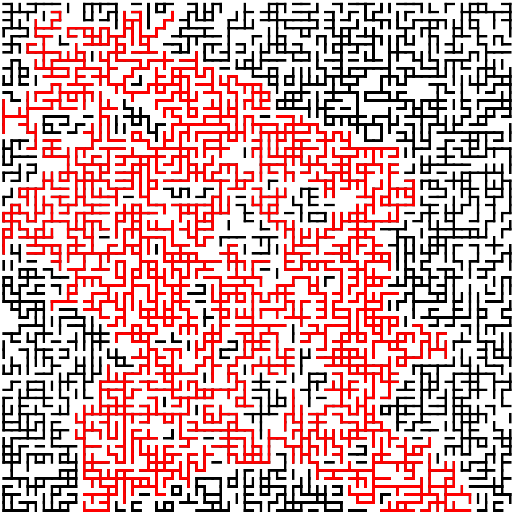

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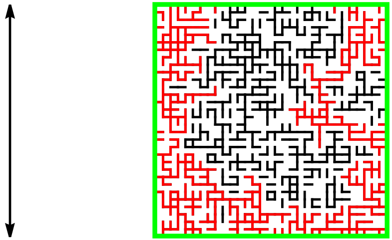
Typical sizes (or lack thereof) revealed by *coarsening*:

1. Zoom out
2. Coarsen
3. Rescale



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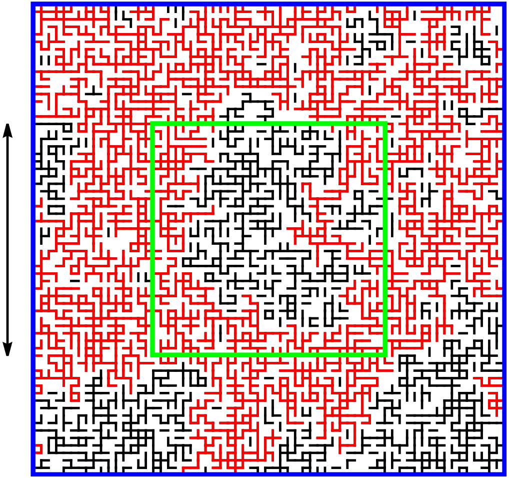

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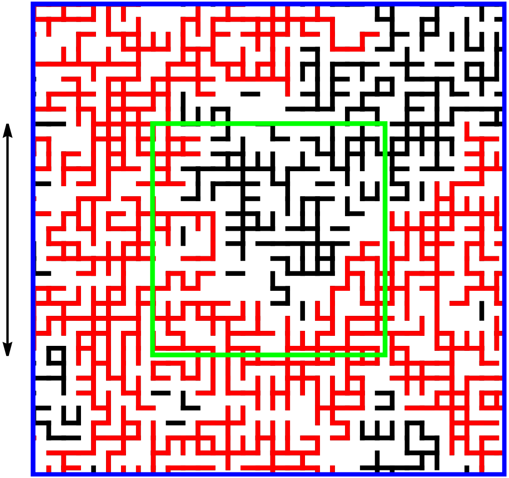

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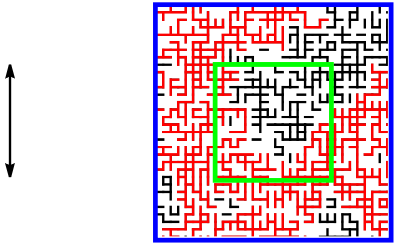

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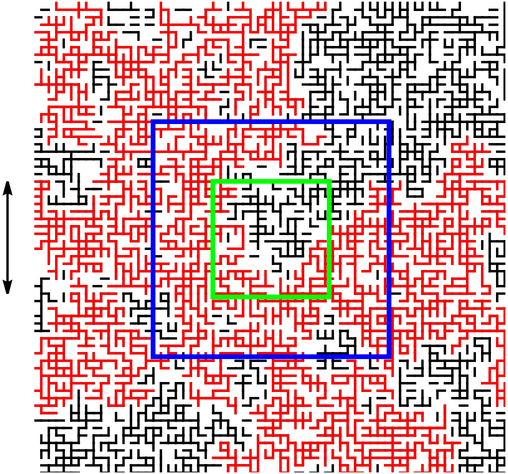

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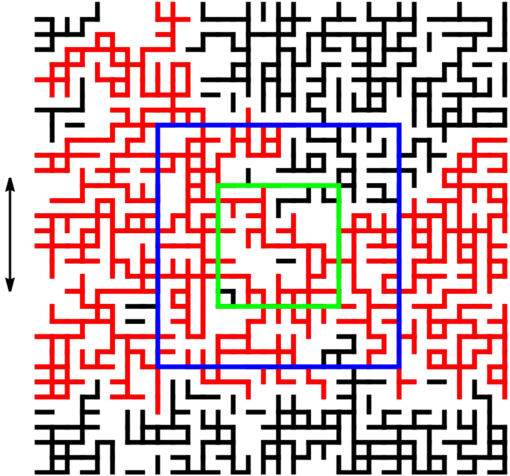

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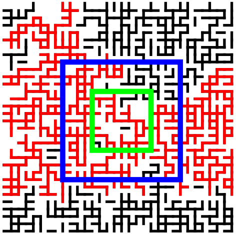

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A canonical example: percolation

Length scale shrinks:

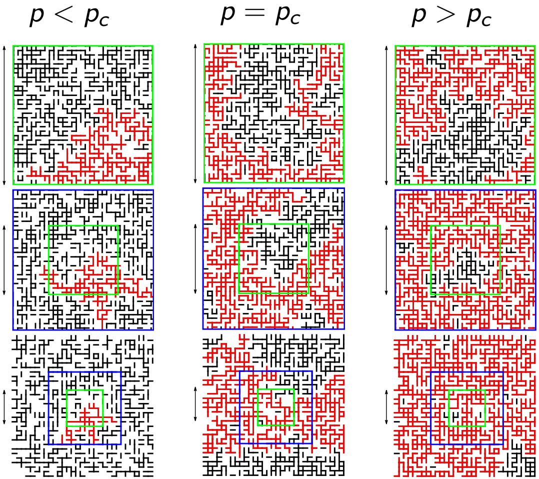
$$\xi \to \frac{1}{2}\xi$$

Difference $\Delta p = p - p_c$ grows:

$$\Delta p \to 2^{1/\nu}\Delta p$$

Invariant combination $\xi \Delta p^{\nu}$ stays the same:

$$\xi \Delta p^{\nu} \to (\xi/2)(2^{1/\nu}\Delta p)^{\nu} = \xi \Delta p^{\nu}$$



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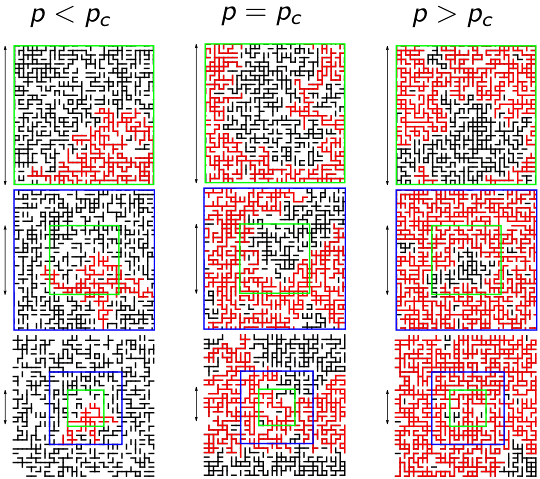
$$\xi \Delta p^{\nu} = C \qquad \Longrightarrow \qquad \xi = C \Delta p^{-\nu}$$



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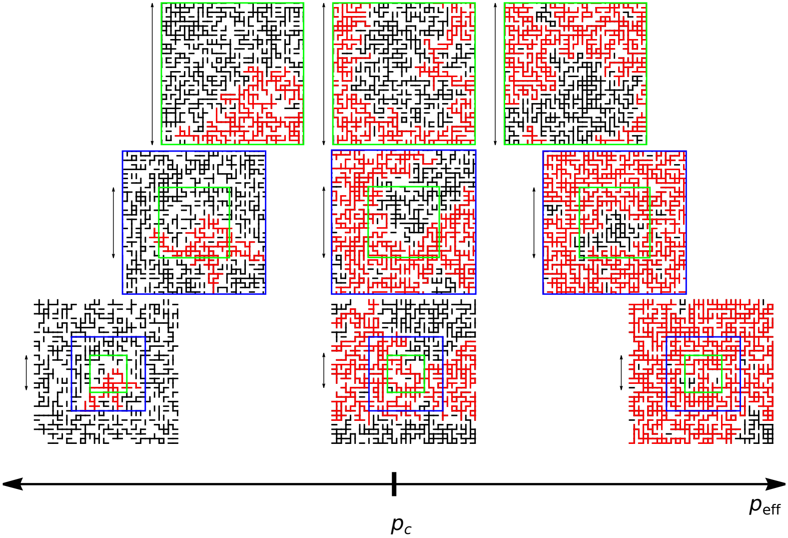
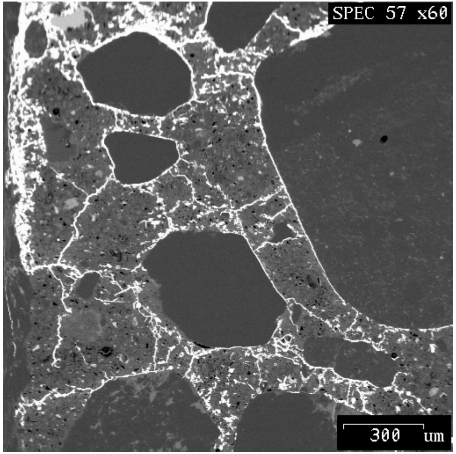


Quasibrittle fracture

Introduction

A crack in the concrete of the Clark Hall stairwell.
Jaron Kent-Dobias, unpublished (2019).



SEM image of stress-induced microcracks in concrete.
Kamran M. Nemati, Scanning 19 **6** 426–430 (1997).

Quasibrittle fracture

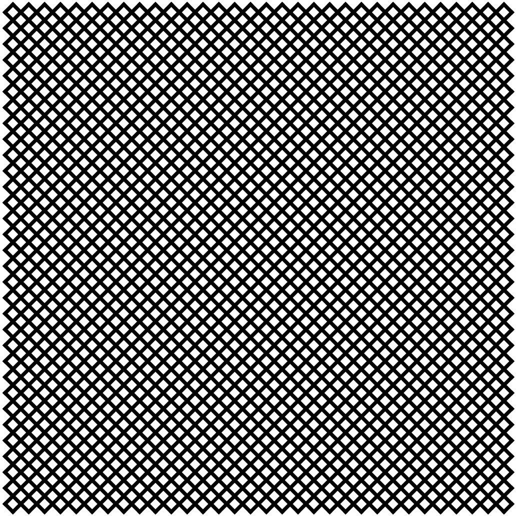
The random fuse model

Fuse: a resistor with a failure threshold

Network of fuses with random thresholds

Disorder controlled by β:

- Large β means *small disorder*
- Medium β means *medium disorder*
- Small β means *high disorder*



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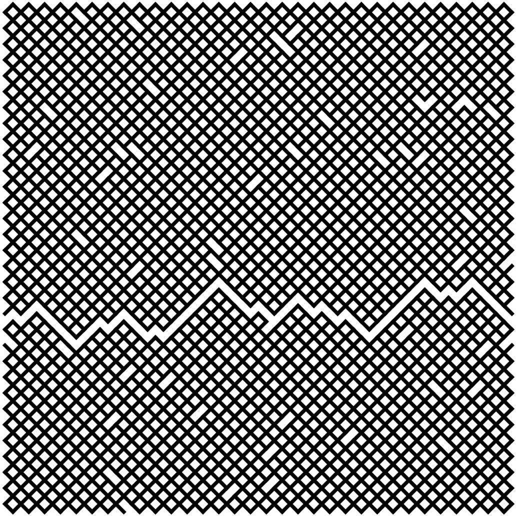

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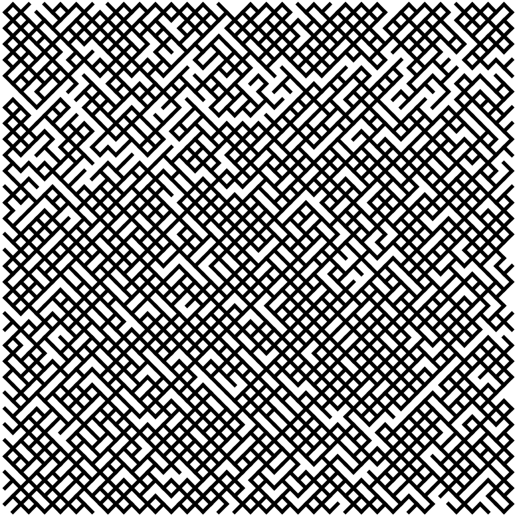

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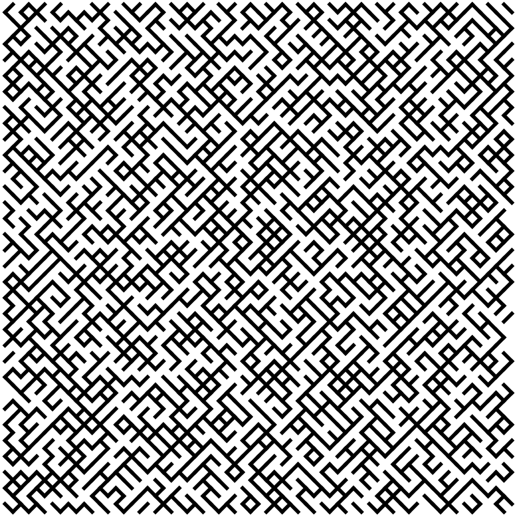


Quasibrittle fracture

High disorder and percolation

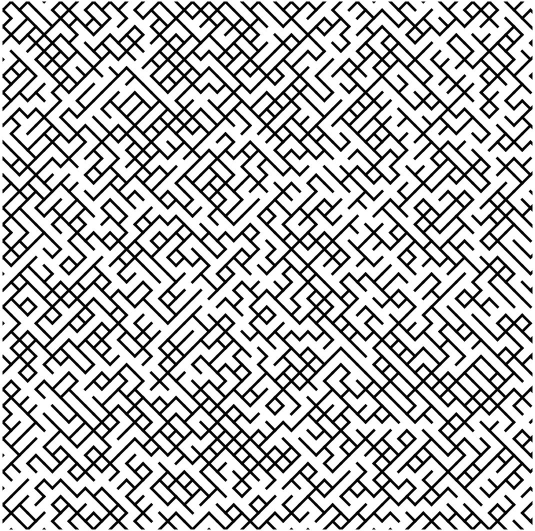
Infinite disorder resembles percolation.

Coarsening moves farther from p_c *and* in other directions:

$$\Delta p \to 2^{1/\nu} \Delta p \qquad \beta \to 2^{\alpha} \beta$$

Infinite size with finite disorder breaks at infinitesimal current!

$$L \to \frac{1}{2} L$$



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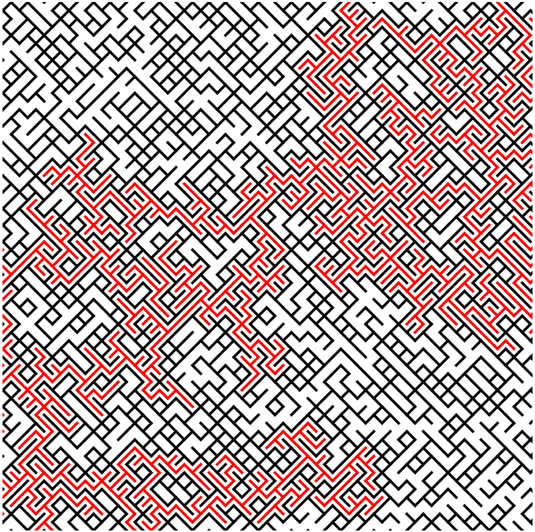

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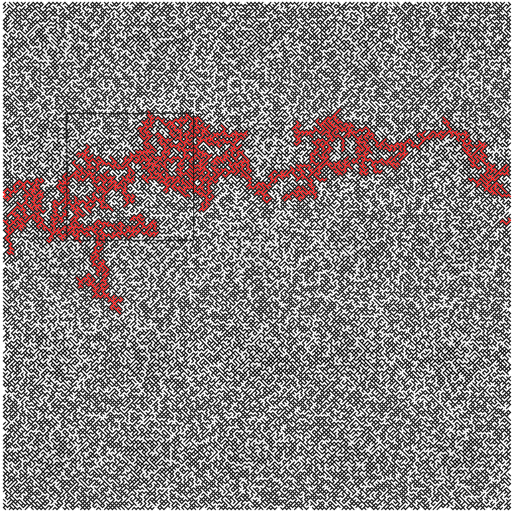

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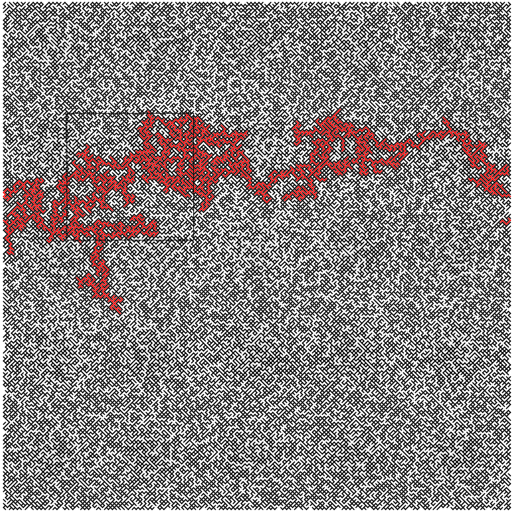
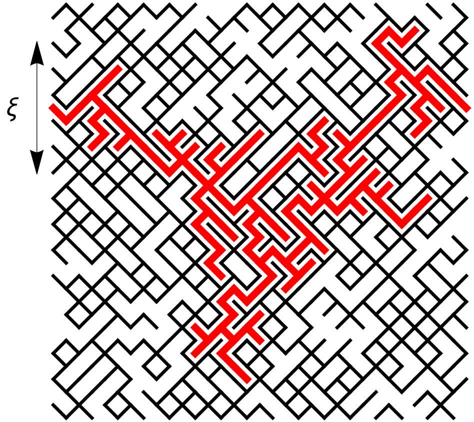
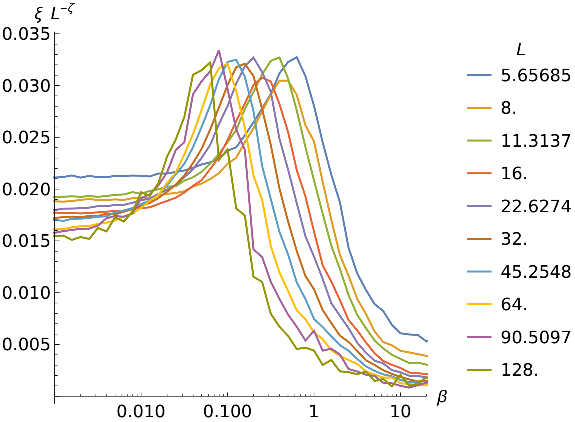


Quasibrittle fracture

Evidence of percolation crossover from finite-size scaling





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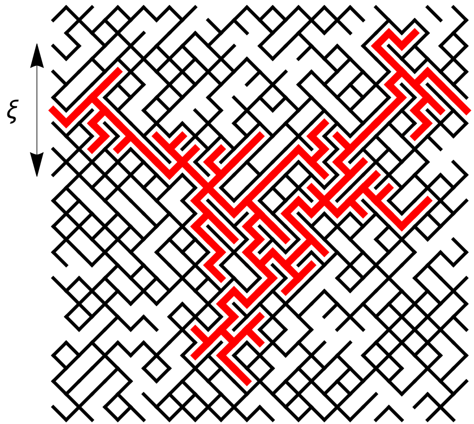
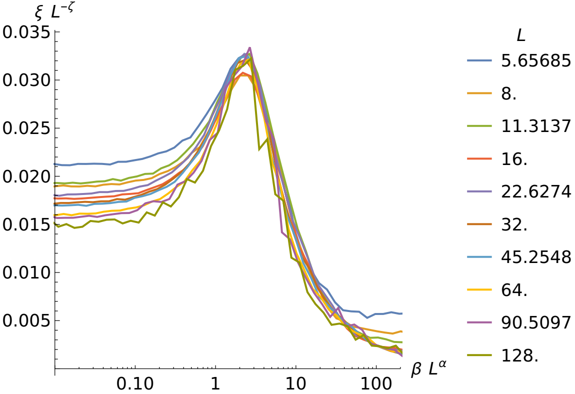
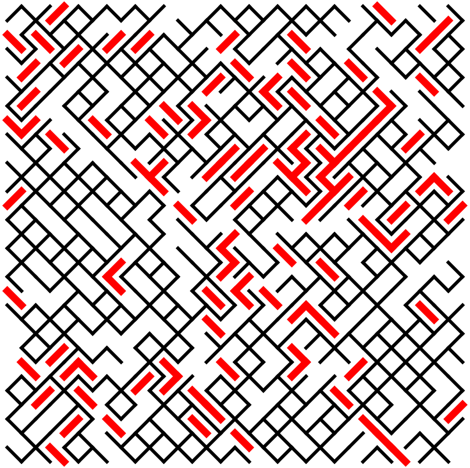


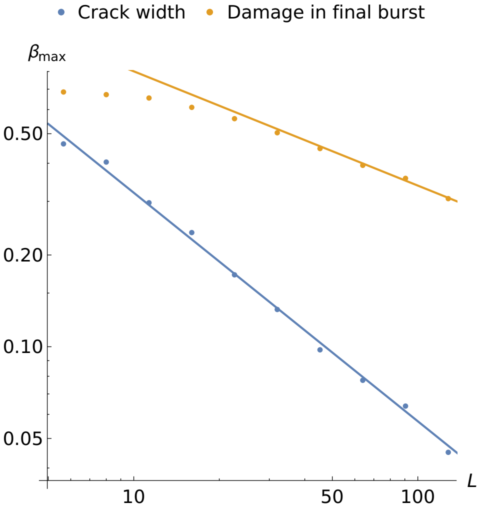


Quasibrittle fracture

Issues with crossover in dynamic quantities



Fuses broken in the final burst that severed the network



Quasibrittle fracture

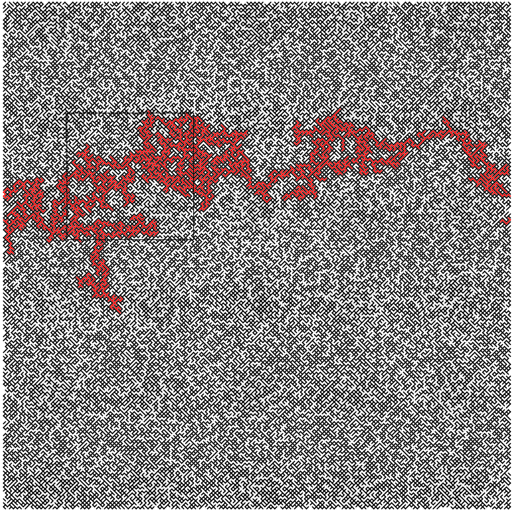
Summary & conclusions

Structural properties of quasibrittle cracks governed by crossover from percolation

Singular dynamic properties not easily explained by same scaling

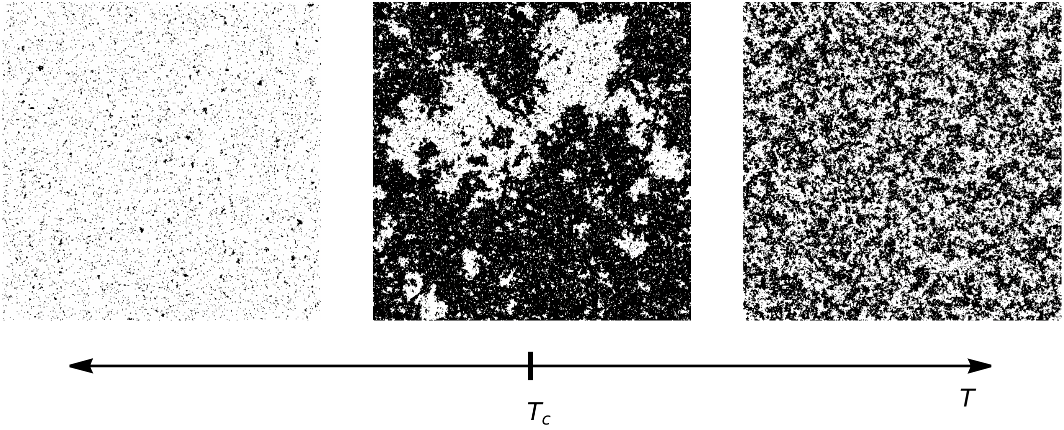
Outstanding theories:

- Novel behavior in transition line for large β
- Second fixed point at large β governs singular dynamics



Essential singularities at abrupt transitions

Coarse graining magnetic models



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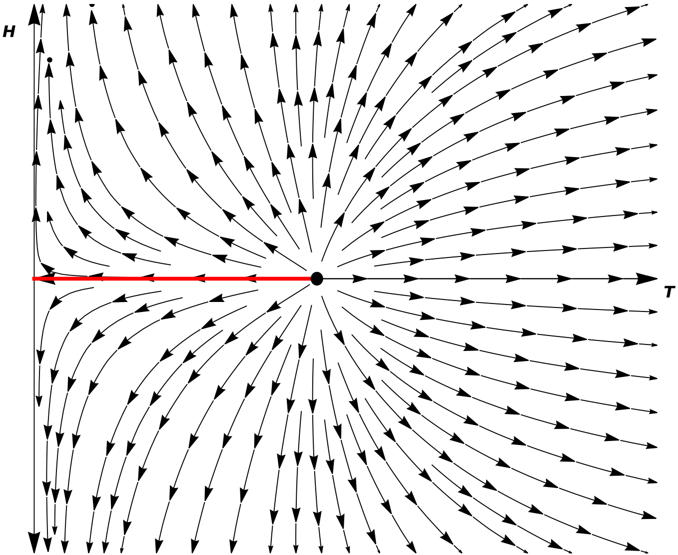

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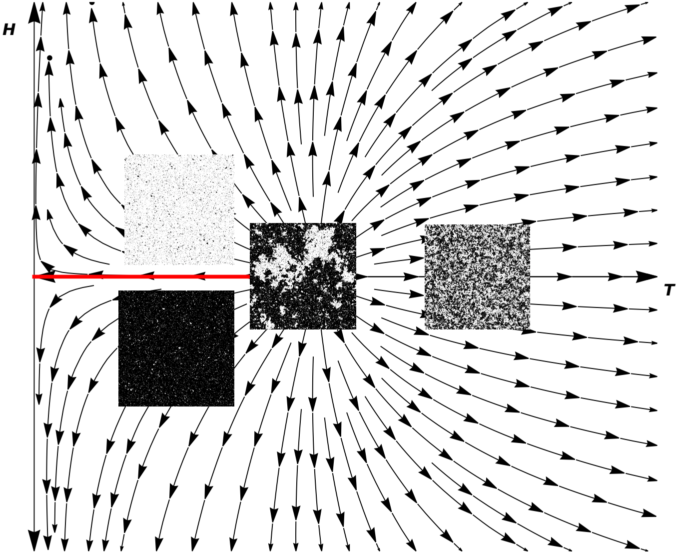

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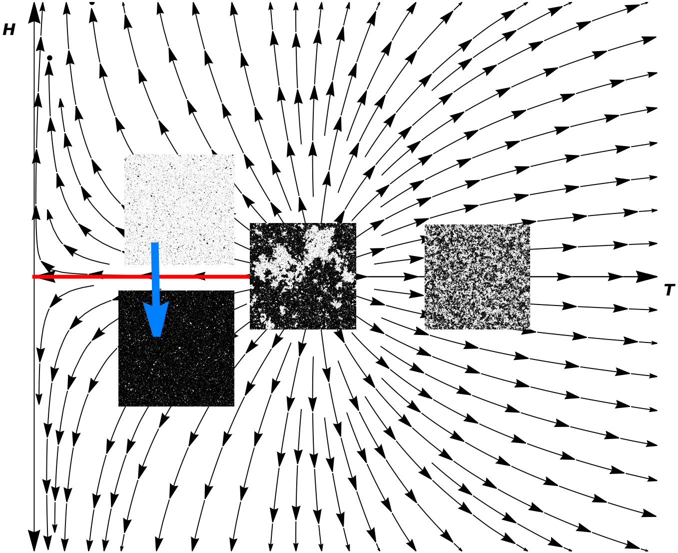

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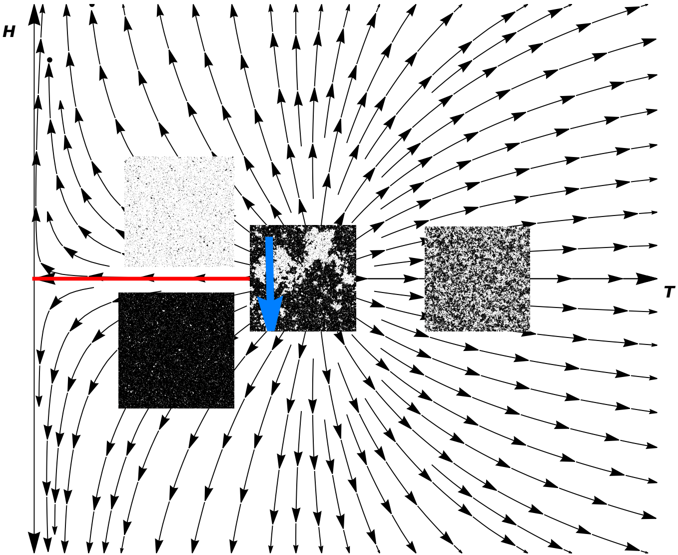


Essential singularities at abrupt transitions

The metastable state and droplets

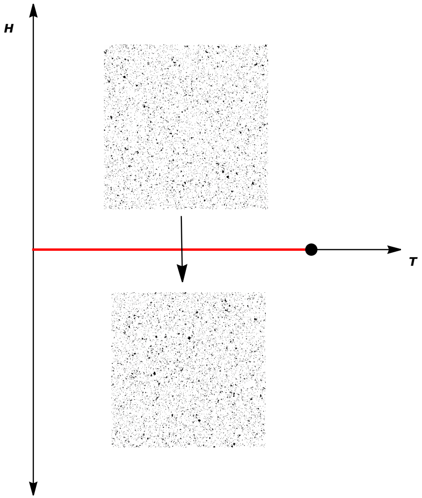
Metastability is common!

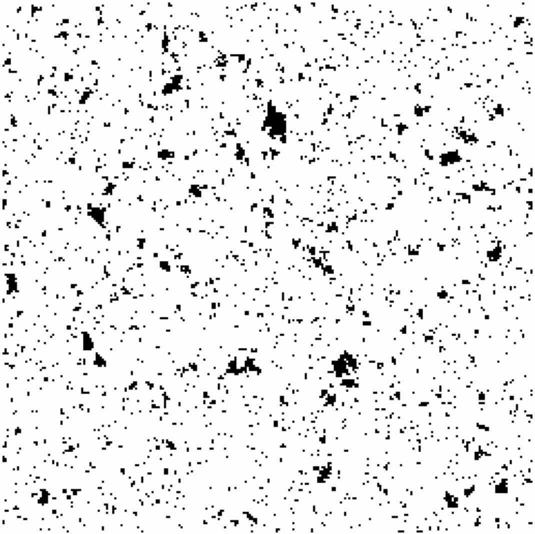
Supercooled Water – Right Before Your Eyes, DrDIYhax, YouTube (2017)

Super heated water can explode outside of microwave, New York Post, YouTube (2017)

Essential singularities at abrupt transitions

The metastable state and droplets





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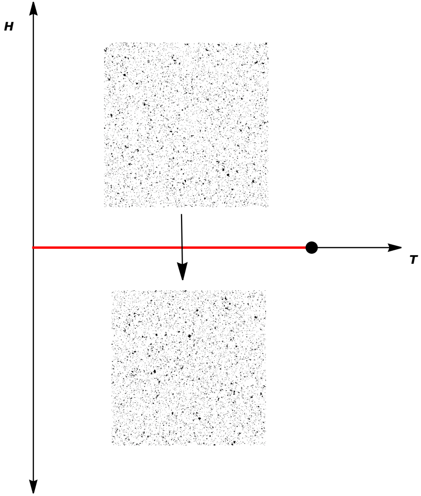
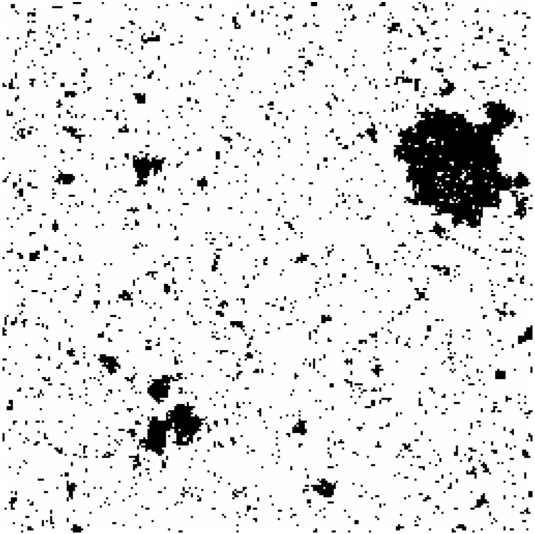




Essential singularities at abrupt transitions

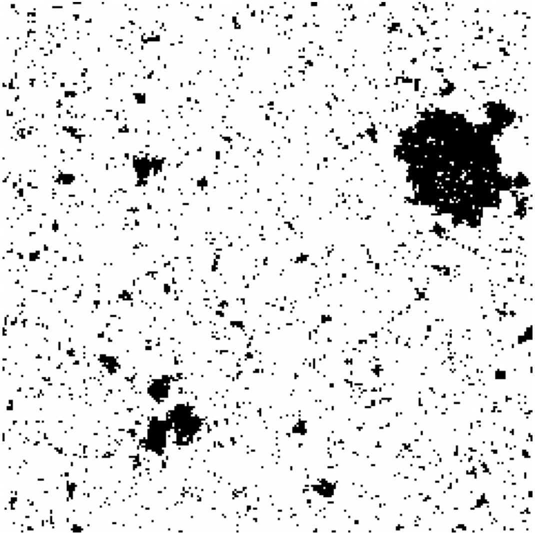
Droplets & decay rate

Surface energy cost $\propto 2\pi r \times \Sigma$

Bulk energy gain $\propto \pi r^2 \times \Delta M \, |H|$

At $r_c \propto |H|^{-1}$, bulk gain beats surface cost

Cost $\Delta E_c \propto |H|^{-1}$ diverges as $H \to 0$



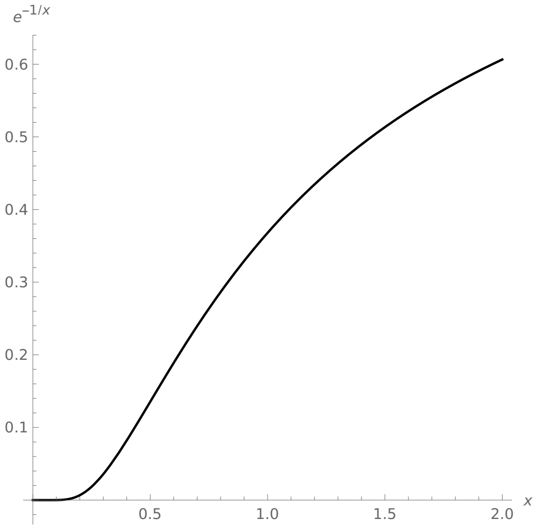
Essential singularities at abrupt transitions

Decay rate & imaginary free energy

Decay rate given by Arrhenius law
$\Gamma \propto e^{-\Delta E_c/T} \sim e^{-B/|H|}$

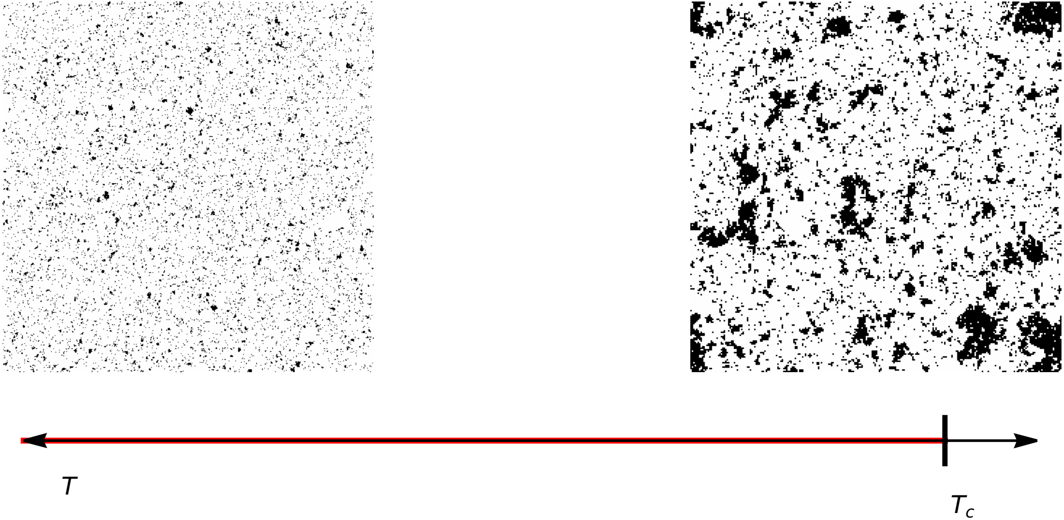
Decay relates to *imaginary free energy*
$\operatorname{Im} F \propto \Gamma$.

$$\operatorname{Re} F = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Im} F(H')}{H - H'} \, dH'$$



Essential singularities at abrupt transitions

Describing critical behavior



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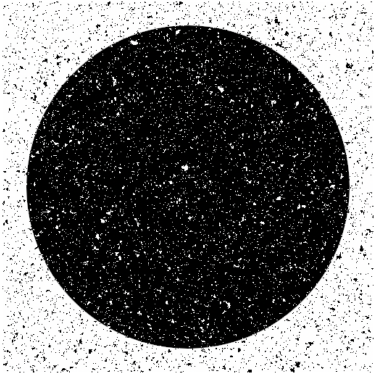

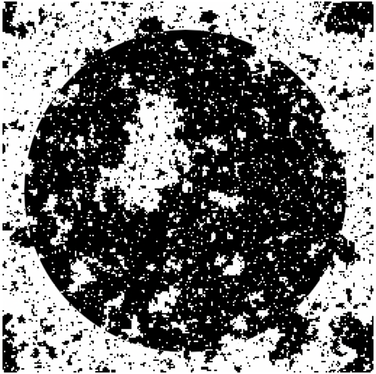

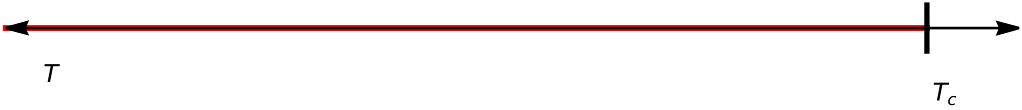

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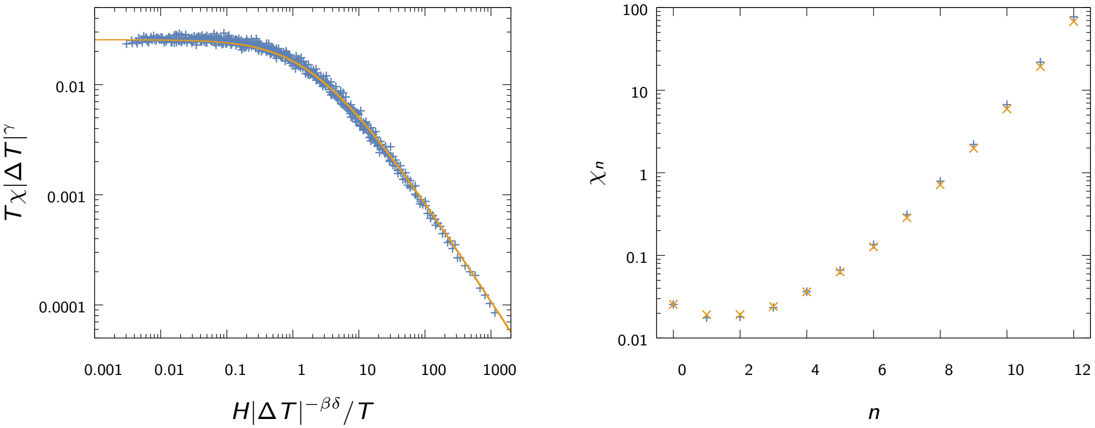


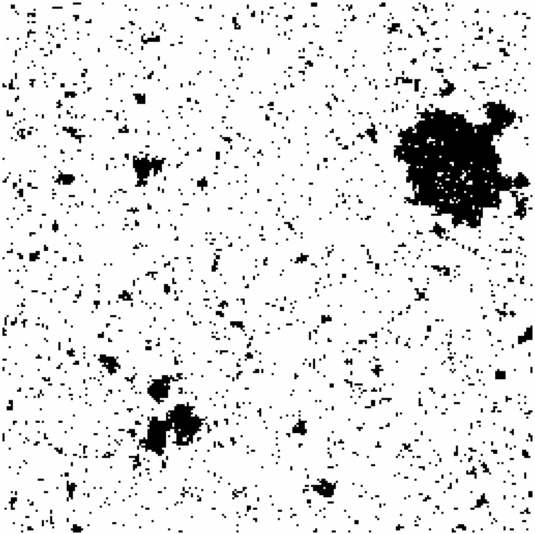
Essential singularities at abrupt transitions

Conclusions

Naïve droplet model closely describes critical singularity in 2D Ising model

Developing ways to incorporate this singularity in iterative approximation

Works less well for 3D Ising—why?



Cluster algorithms for lattice models in fields

Simulating critical lattice models is slow

Timescales diverge at critical points

Realistic local methods are slow

Cluster methods are much faster

Don't naturally work with on-site potentials like external fields

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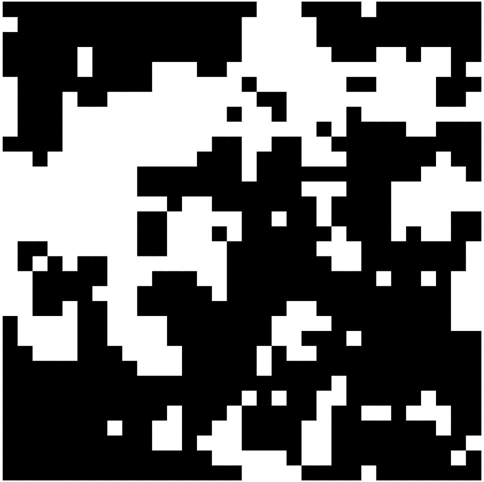

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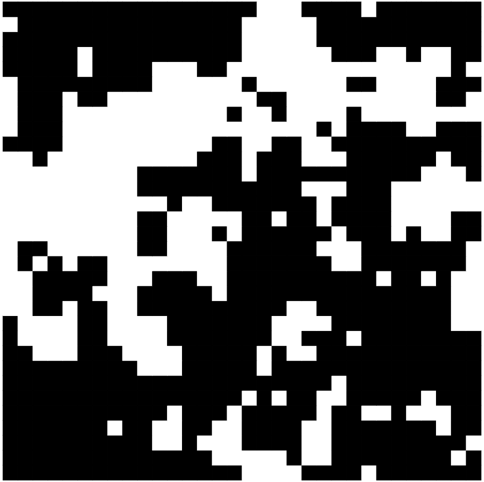

Cluster algorithms for lattice models in fields

Cluster methods without potentials

With

$$p_{ij} = \begin{cases} 1 - e^{\beta \Delta E_{ij}} & \Delta E_{ij} > 0 \\ 0 & \text{otherwise,} \end{cases}$$

1. Pick a reflection
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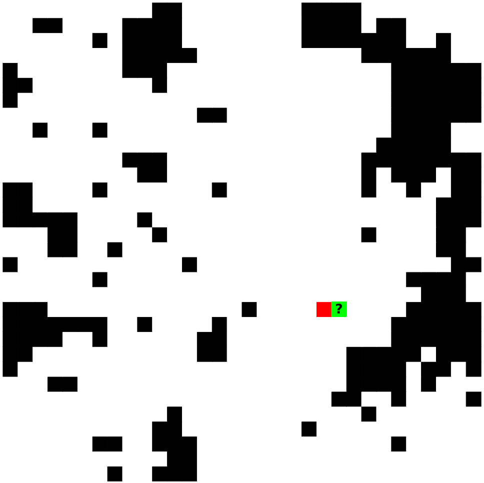

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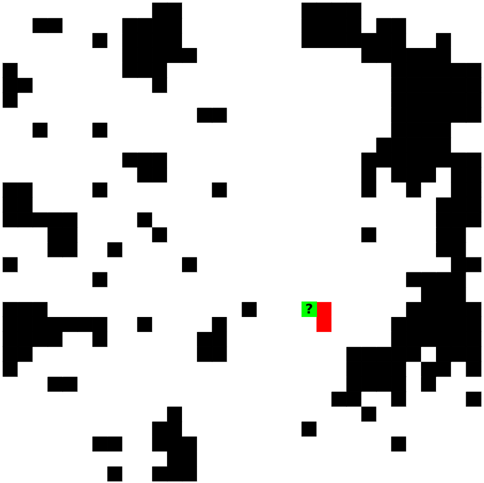

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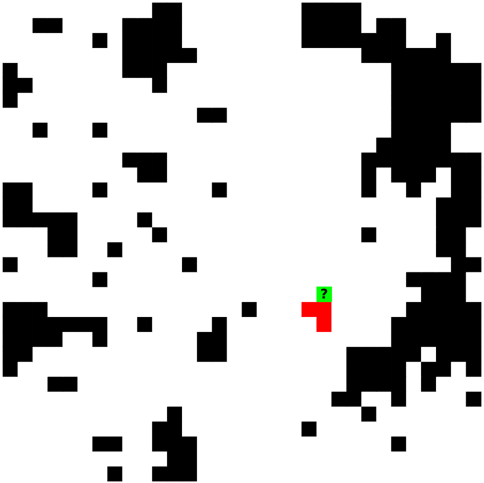

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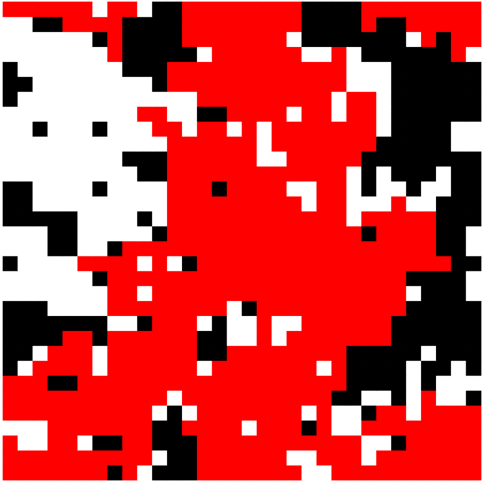

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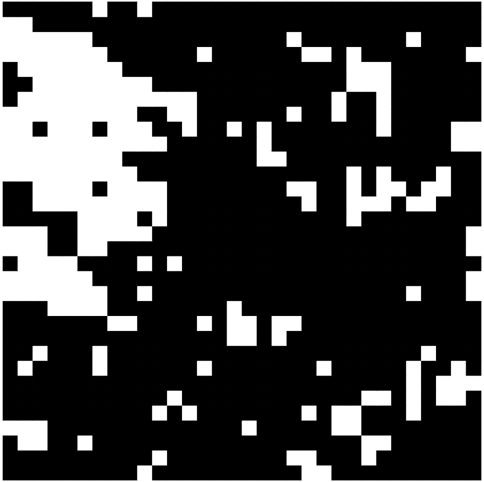
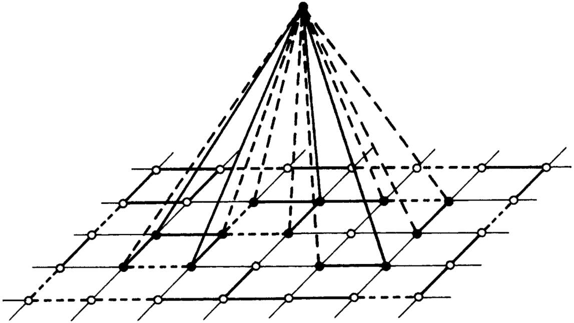


Cluster algorithms for lattice models in fields

The ghost site representation



Coniglio, de Liberto, Monroy, & Peruggi. J Phys A **22** (1989) L837

Introduce new site adjacent to all others, give it funny coupling

Degree of freedom on site is a *symmetry group element*, not a spin

Cluster algorithms for lattice models in fields

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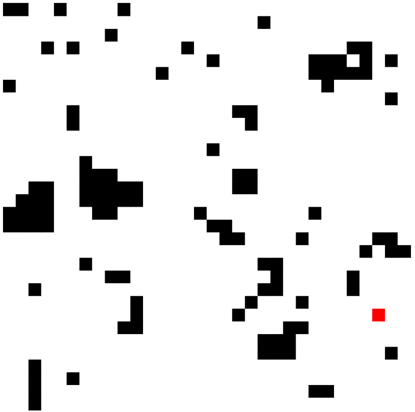

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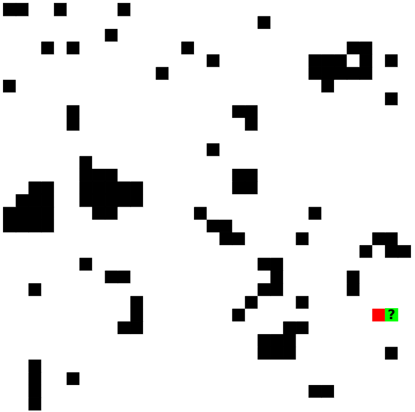

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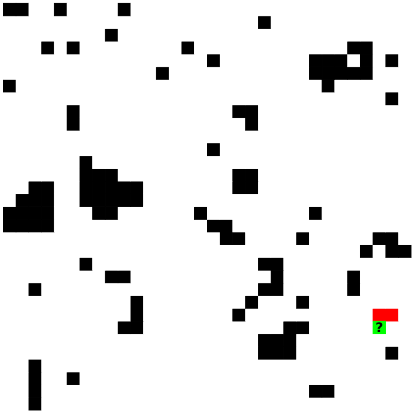

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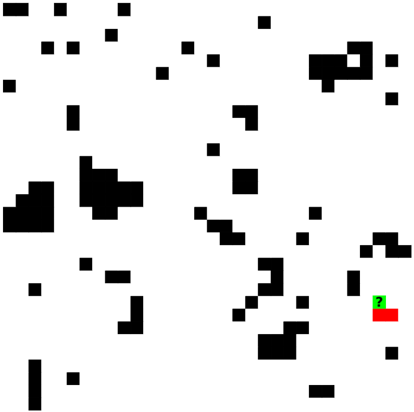

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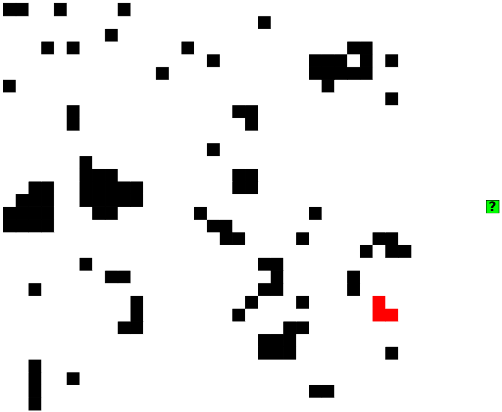

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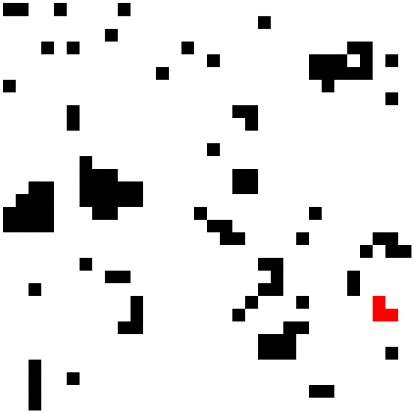

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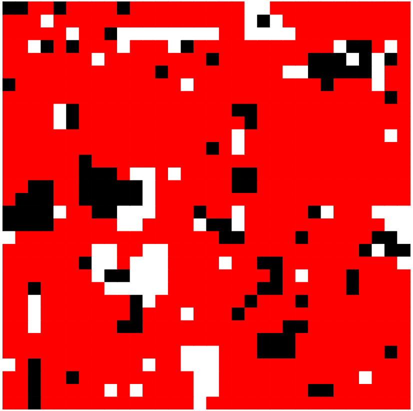

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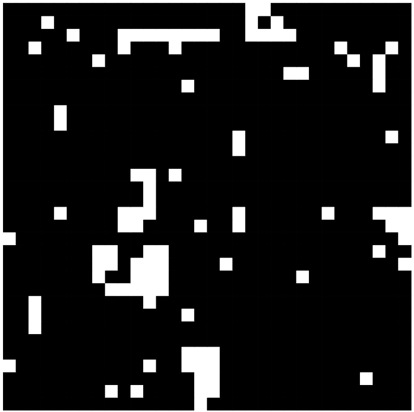

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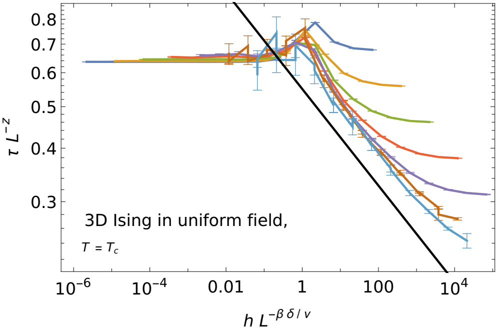
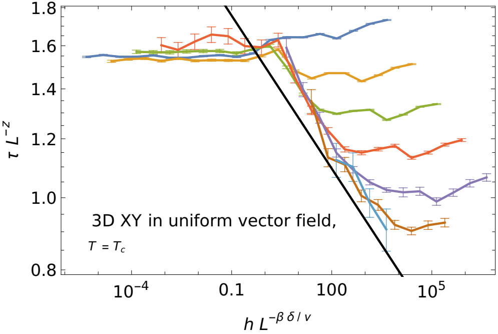
Is it efficient?

Extension is fast: larger field means more efficient

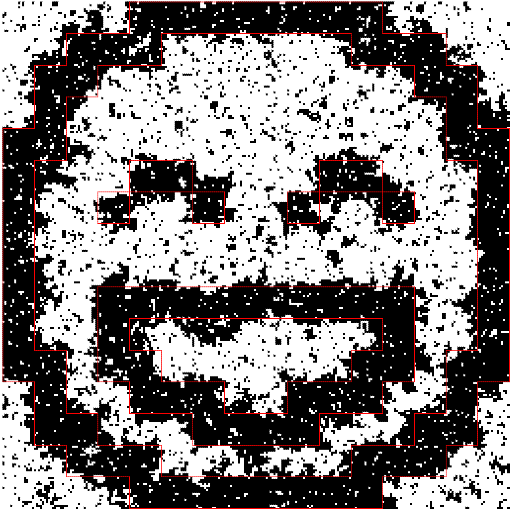
Extension is natural: correlation times scale as predicted by coarsening





Cluster algorithms with background potentials

Introduction

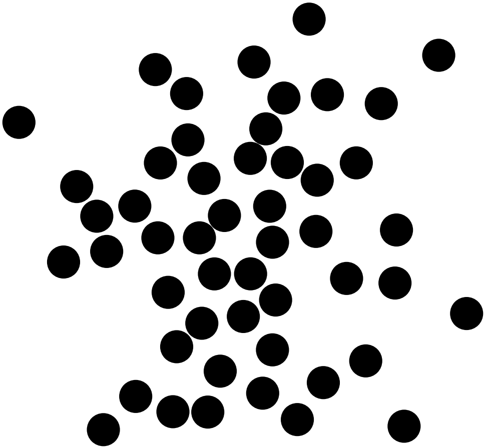


Cluster algorithms with background potentials

Hard spheres without potential

1. Pick a reflection
2. Pick a seed
3. Transform the seed
4. Identify particles with intersections
5. Transform each intersecting particle
6. Repeat 5–6 until exhausted

Dress & Krauth *J Phys A: Math Gen* **28** (1995) 597



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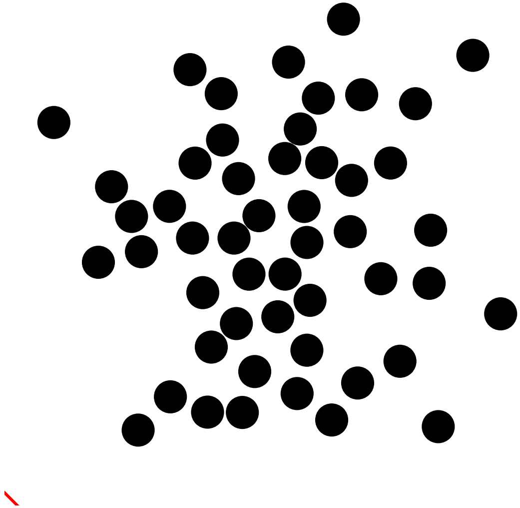

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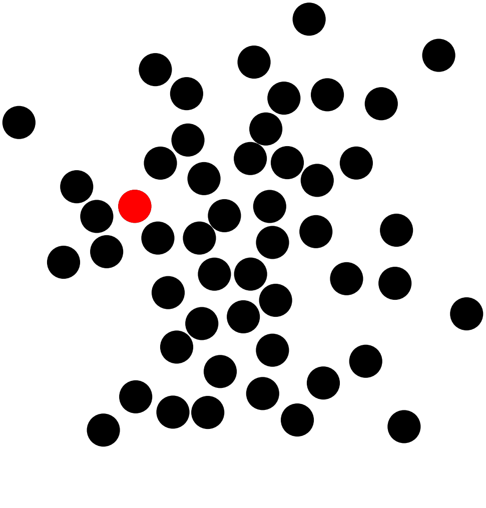

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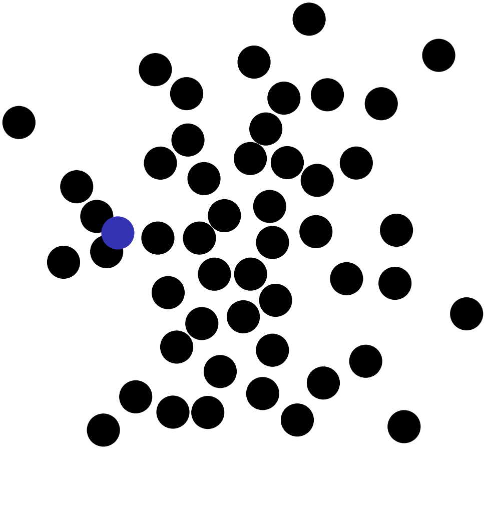

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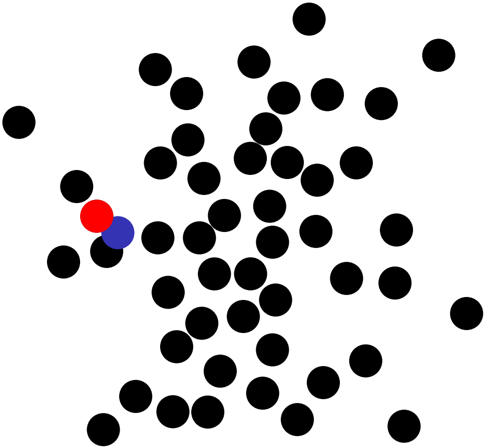

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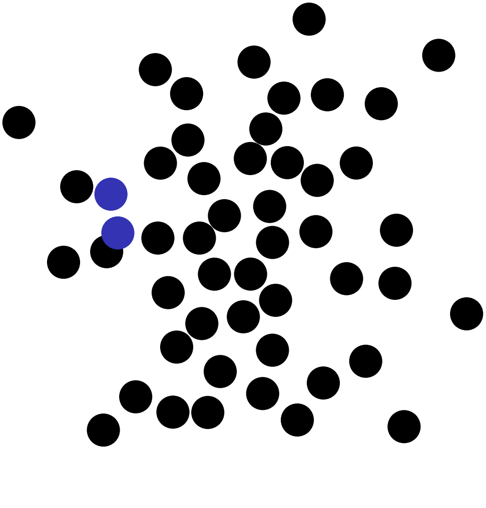

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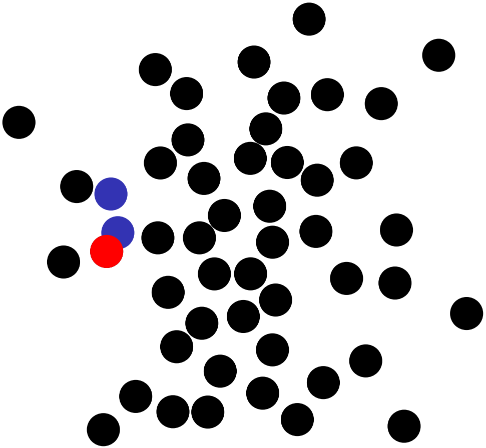

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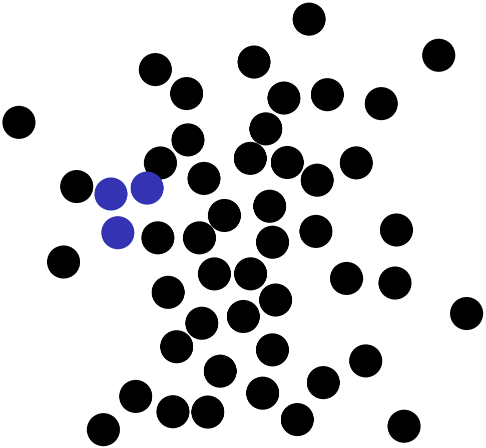

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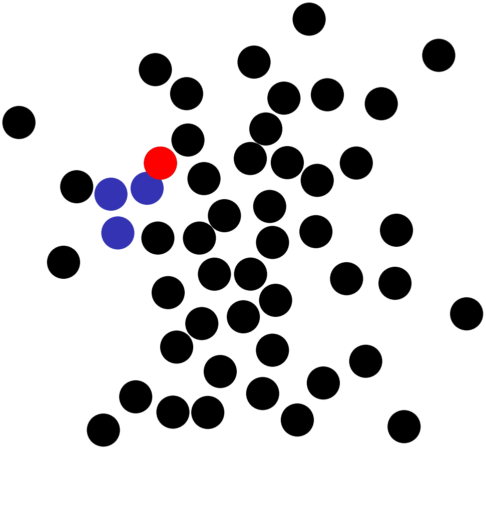

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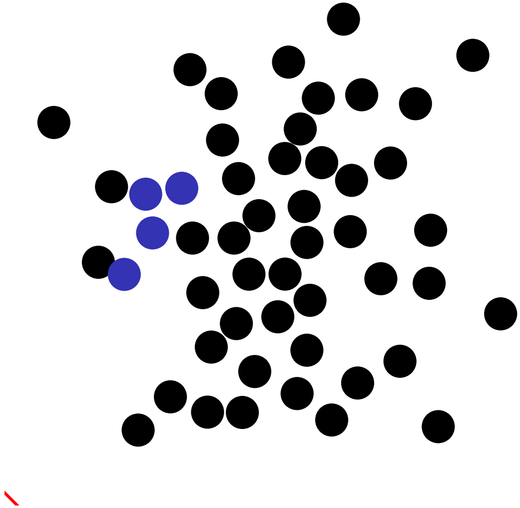

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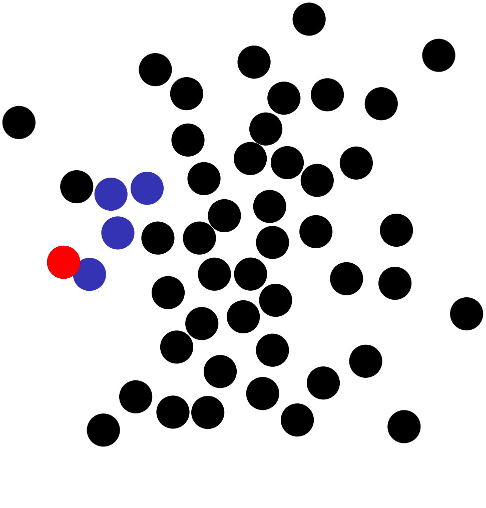

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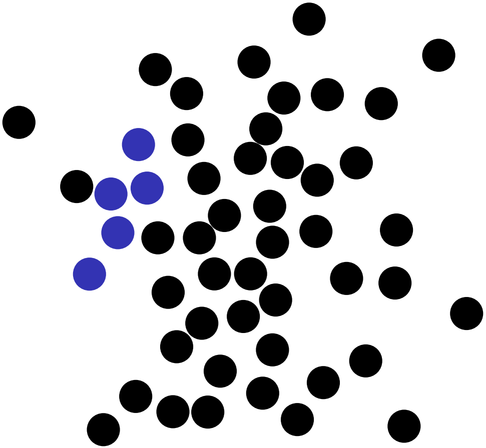

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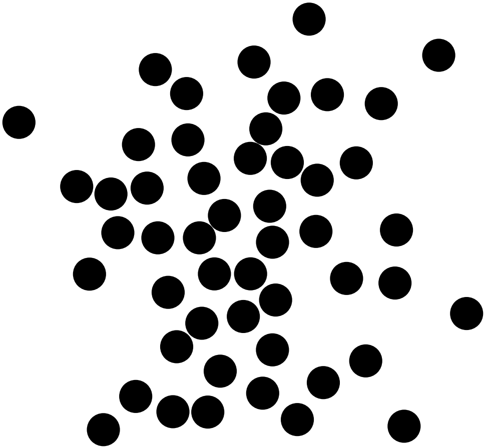

Cluster algorithms with background potentials

Spheres in hard potential

Hard potential? Treat it like a particle!

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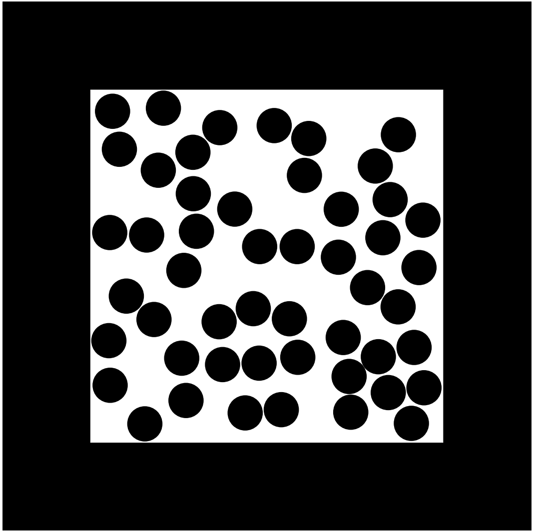

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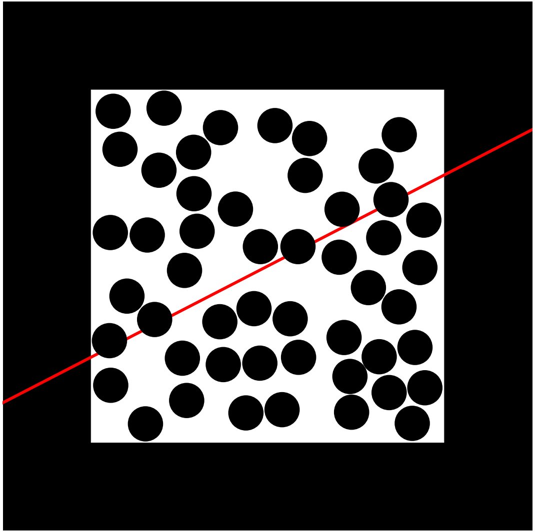

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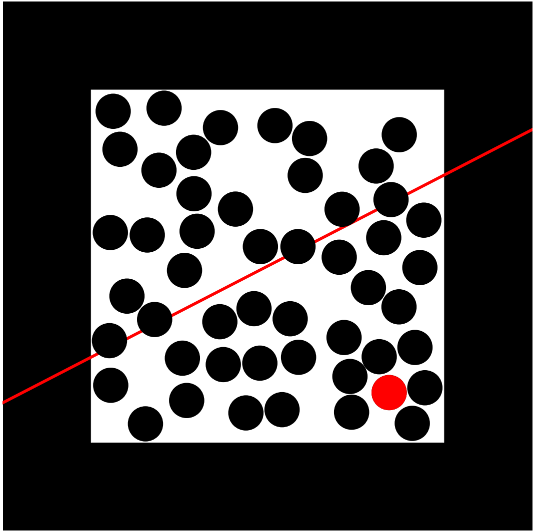

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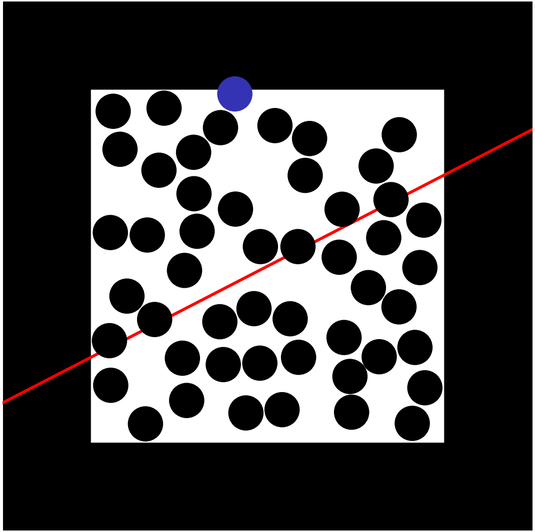

Cluster algorithms with background potentials

Spheres in hard potential

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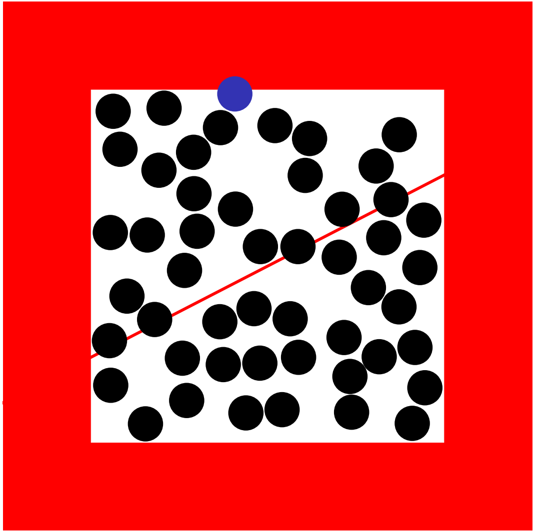

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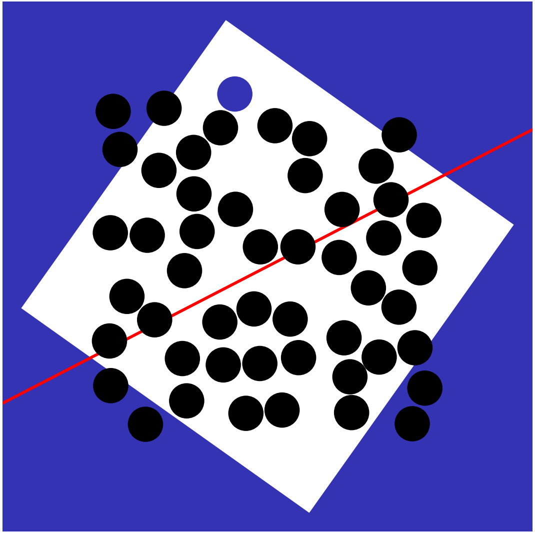

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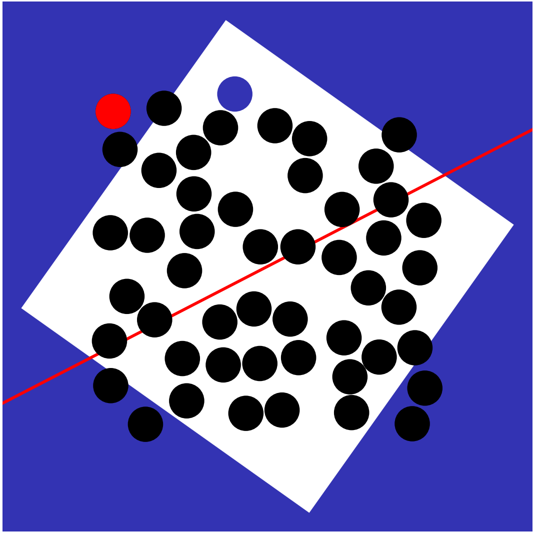

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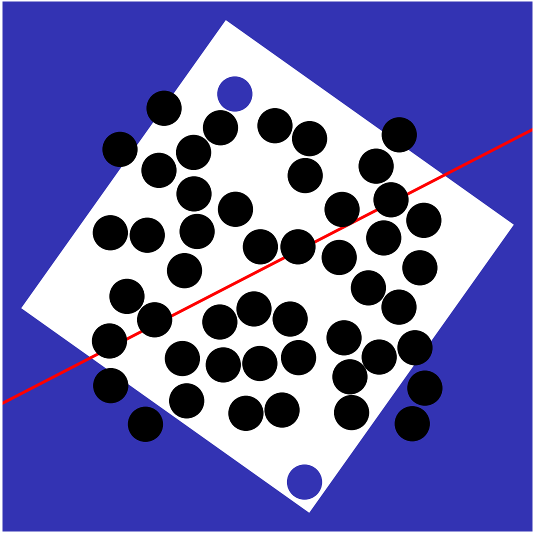

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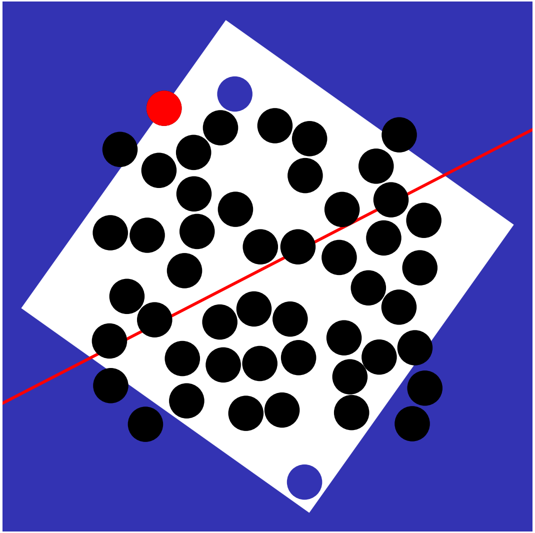

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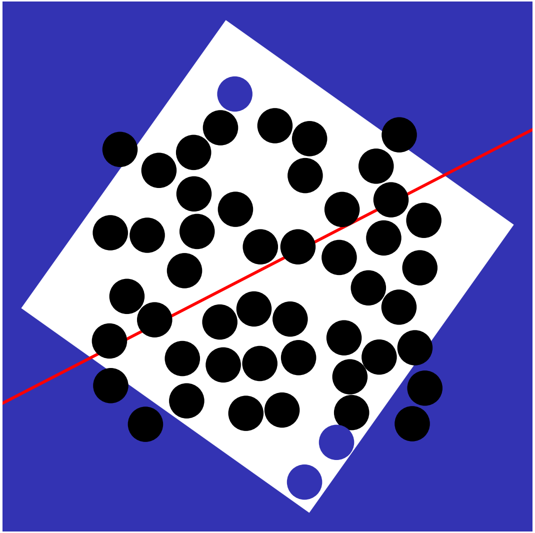

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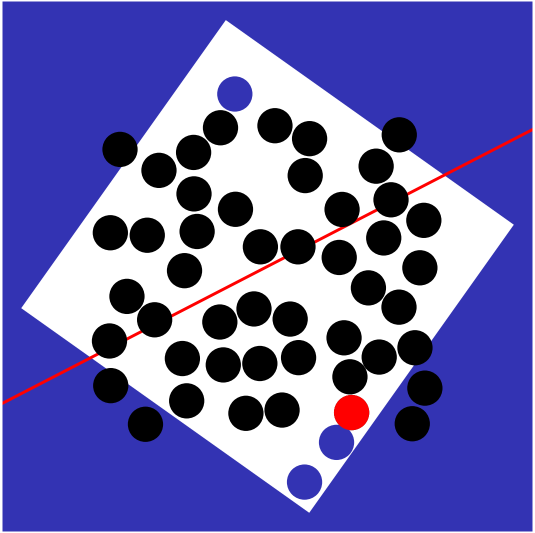

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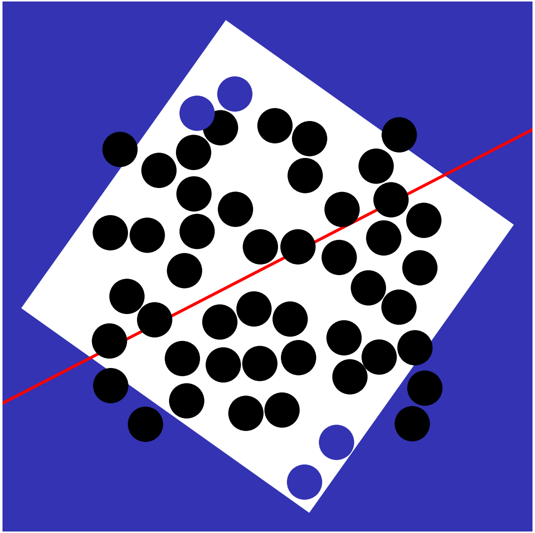

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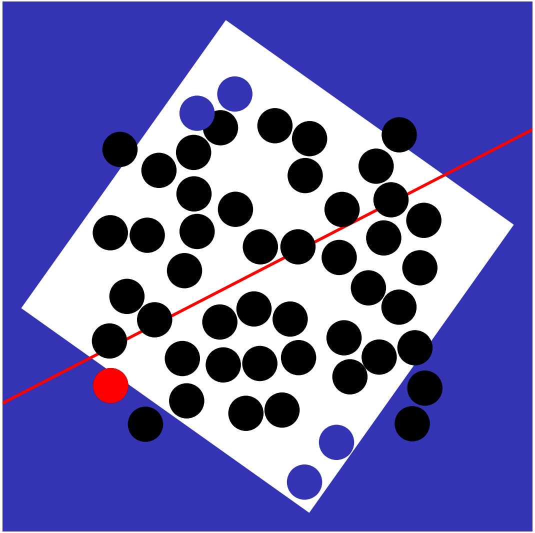

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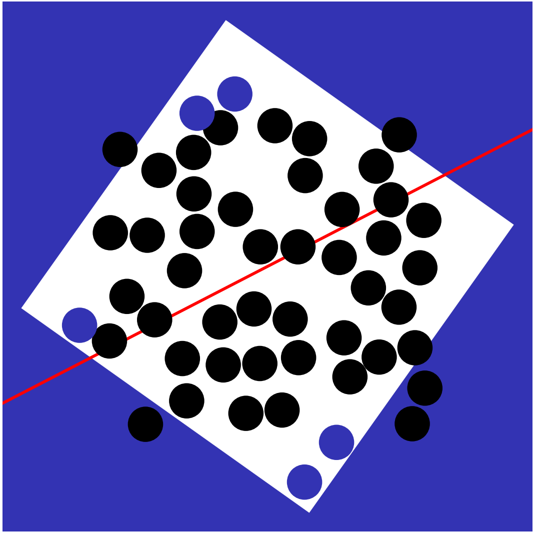

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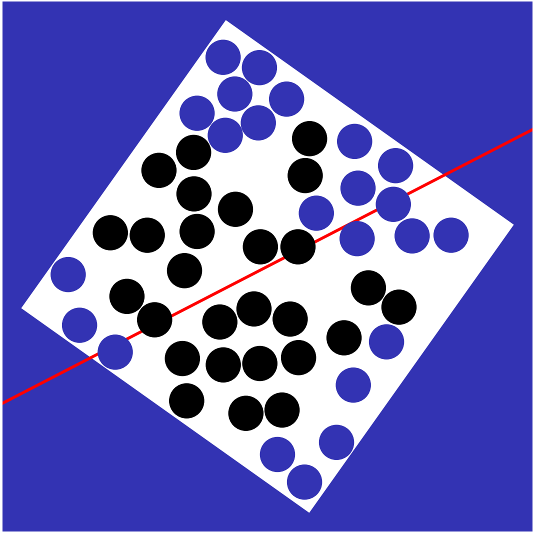

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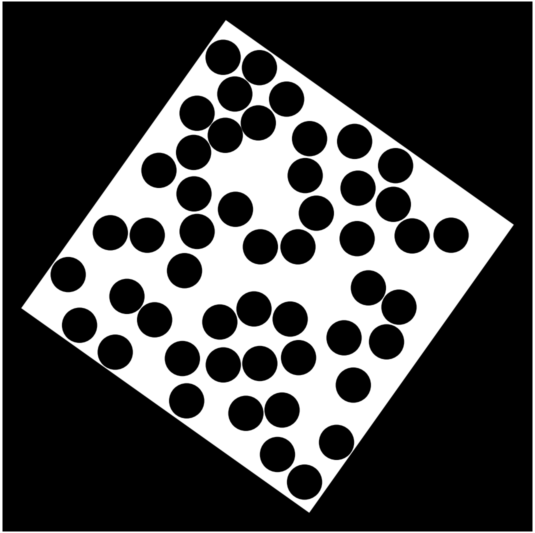
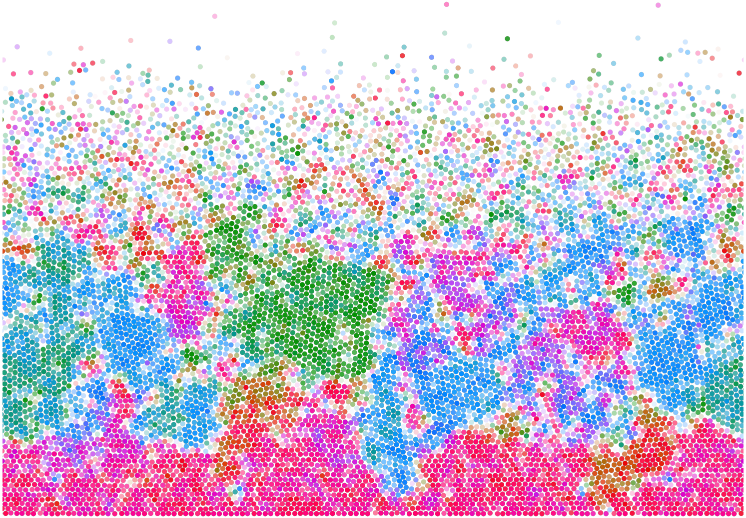


Cluster algorithms with background potentials

Possible applications



Normal forms for infinite-order transitions

Introduction

How do you classify critical phenomena?

Normally, by principal singularity $\xi \sim \Delta p^{-\nu}$

Variety of systems see $\xi \sim e^{-A/\Delta x^{\sigma}}$

Does sharing σ imply shared fixed point?

- 2D XY model
- 2D Coulomb gas
- 2D interfaces
- Percolation in infinite-dimensional growing networks
- Percolation in infinite-dimensional simplicial networks
- 1D inverse-square Ising model
- Kondo model
- 2D sine–Gordon model
- Hexatic–solid transition

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Normal forms for infinite-order transitions

The XY model and the BKT transition

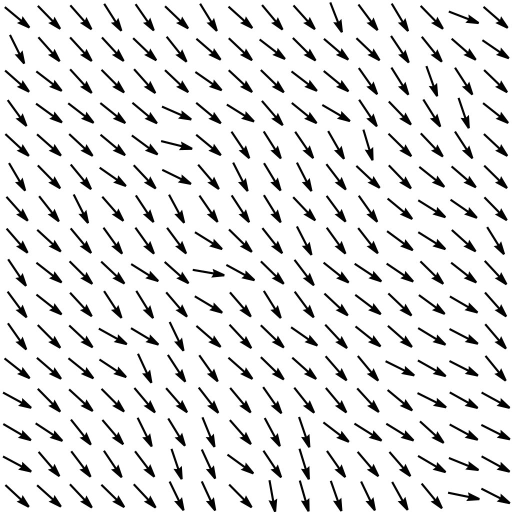
Low-temperature phase is pseudo-long range, vortices are bound

High-temperature phase has unbounded vortices

Unbinding transition governed by Berezinskiĭ–Kosterlitz–Thouless fixed point

For $\Delta x \sim \Delta T$,

$$\xi \propto e^{-A/\sqrt{\Delta x}}$$



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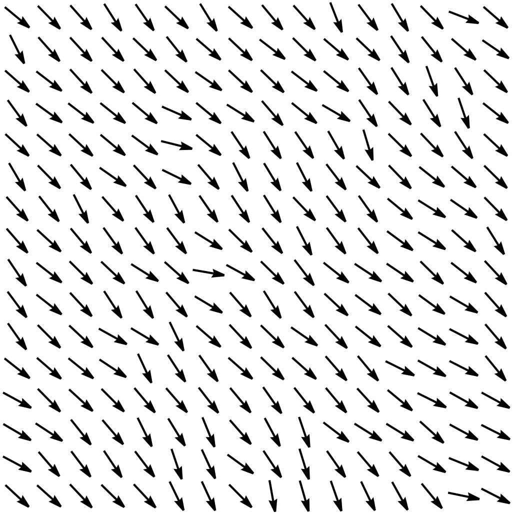

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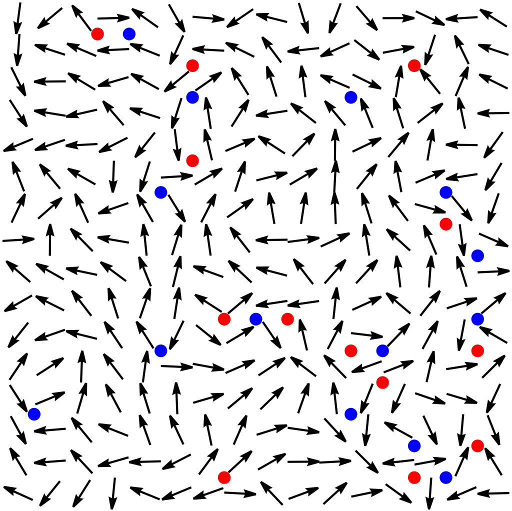

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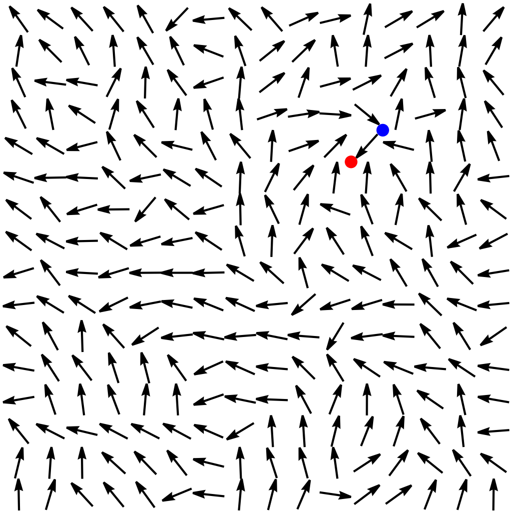


Normal forms for infinite-order transitions

Transitions in growing networks

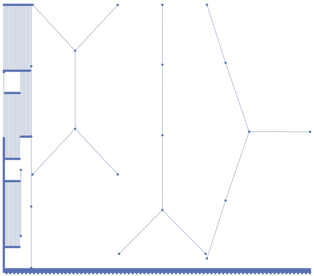
Model of growing networks: every timestep, add a vertex and connect two existing vertices with probability δ.

At $\delta = \delta_c$, infinite cluster emerges with weight

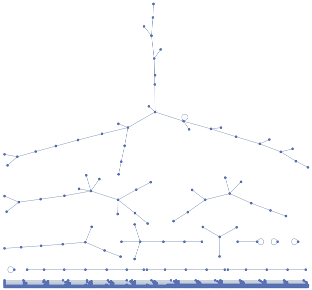
$$S \propto e^{B/\sqrt{\Delta\delta}}$$

Is it BKT?!

$\delta < \delta_c$



$\delta > \delta_c$



Normal forms for infinite-order transitions

How to compare fixed points

Look at differential equations for the coarse graining flow, compare coefficients

Most fixed points: compare truncation

$$\frac{d\Delta p}{d\ell} = \frac{1}{\nu}\Delta p + \cdots$$
$$\frac{d\Delta q}{d\ell} = \frac{1}{\nu'}\Delta q + \cdots$$

If $\nu = \nu' \simeq 0.8774$, probably the same!

Rescaling removes quadratic coefficients for BKT: with $x \sim \Delta T$ and fugacity y,

$$\frac{dx}{d\ell} = -y^2 + \cdots$$
$$\frac{dy}{d\ell} = -xy + \cdots$$

$\sigma = \frac{1}{2}$ in $\xi \sim e^{A/\Delta x^{\sigma}}$ depends only on truncation!

Normal forms for infinite-order transitions

How to be BKT

Higher order rules for BKT:

- *Neutrality*: symmetry in $y \to -y$.
- *Triviality*: at $y = 0$, nothing flows.

$$\frac{dx}{d\ell} = -y^2 + a_1 x y^2 + \cdots$$
$$\frac{dy}{d\ell} = -xy + a_2 y^3 + a_3 x^2 y + \cdots$$

Smooth coordinate transformation changes cubic coefficients:

$$\tilde{x} = x + X_1 x^2 + X_2 xy + X_3 y^2$$
$$\tilde{y} = y + Y_2 xy + Y_3 y^2$$

One can define $\tilde{x}$, $\tilde{y}$ such that

$$\frac{d\tilde{x}}{d\ell} = -\tilde{y}^2 - b_0 \tilde{x}\tilde{y}^2 + \cdots$$
$$\frac{d\tilde{y}}{d\ell} = -\tilde{x}\tilde{y}$$

for *universal* b_0. Appears in corrections to scaling:

$$\xi \propto e^{-\pi/\sqrt{\Delta\tilde{x}}}\left(1 - \frac{\pi b_0^2}{12}\sqrt{\Delta\tilde{x}} + \cdots\right)$$

Normal forms for infinite-order transitions

How to be BKT

With same quadratic form but no neutrality constraint, cannot bring equations to same simplest form:

$$\frac{d\tilde{x}}{d\ell} = -\tilde{y}^2 - c_1\tilde{y}^3 + \cdots$$
$$\frac{d\tilde{y}}{d\ell} = -\tilde{x}\tilde{y} - c_2\tilde{y}^3$$

Systems with BKT-like singularity but no neutrality *cannot* be BKT!

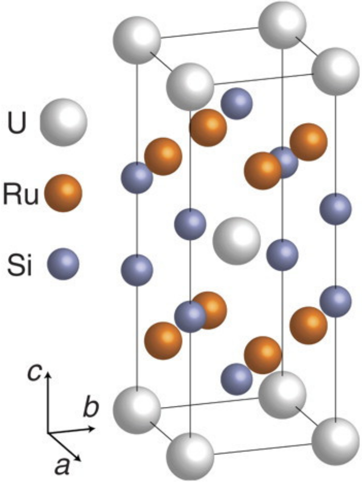
Do growing networks share the same subleading singularity? We don't know yet.

Exploring use of analytic methods & finite-size scaling.

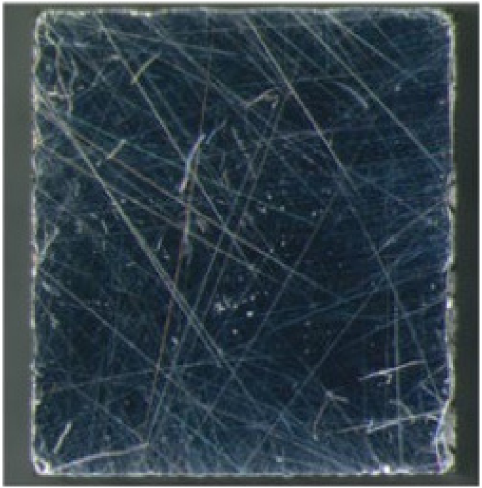
Looking to other so-called BKT transitions in, e.g., the Kondo problem, 1D inverse-square Ising, simplicial percolation.

Hidden Order in URu_2Si_2

Introduction



Okazaki *et al.*, Science 331 6016 439–442 (2011)

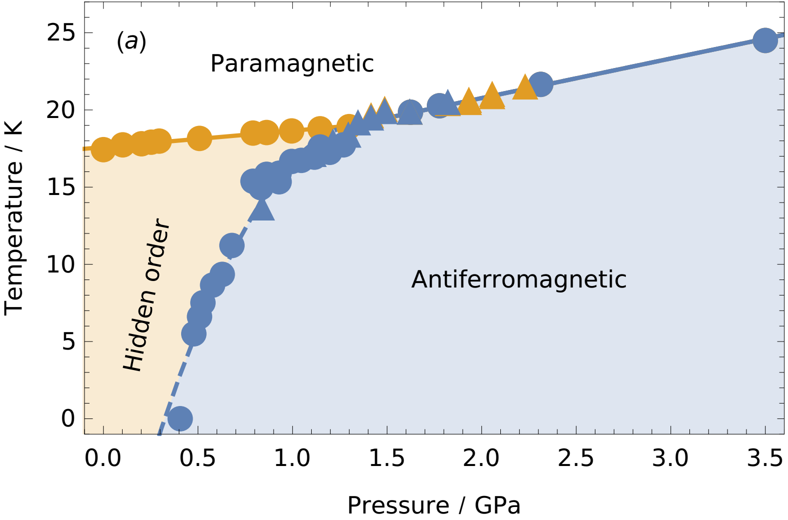


Ghosh *et al.*, Science Advances 6 10 eaaz4074

Hidden Order in URu_2Si_2

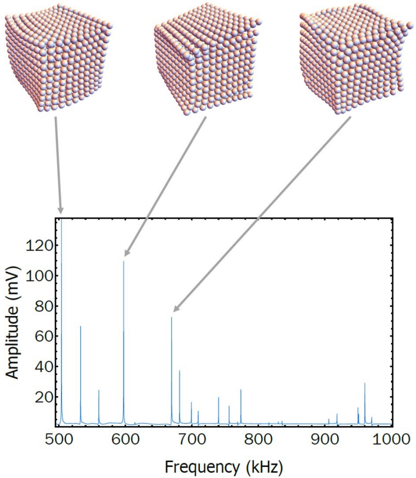
Phase diagram

Data from Hassinger *et al.* Physical Review B **77** 115117 (2008)



Hidden Order in URu_2Si_2

Resonant ultrasound spectroscopy



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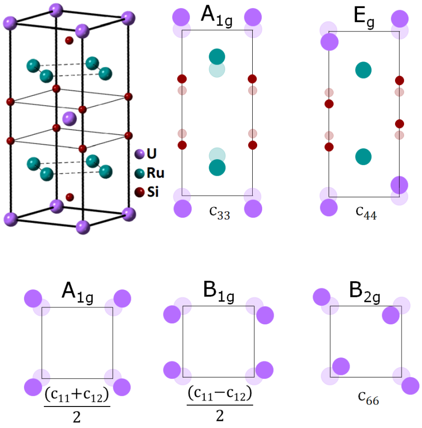
Hidden Order in URu_2Si_2

Strain components and moduli

Ghosh *et al.*, Science Advances 6 10 eaaz4074



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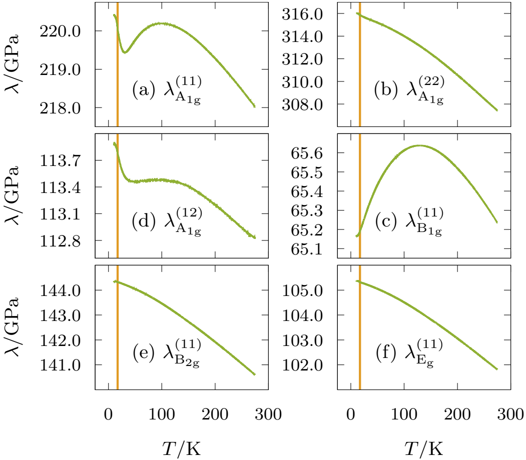

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Lessons from Landau

Vestnik Kavkaza, *Great Baku native Lev Landau*

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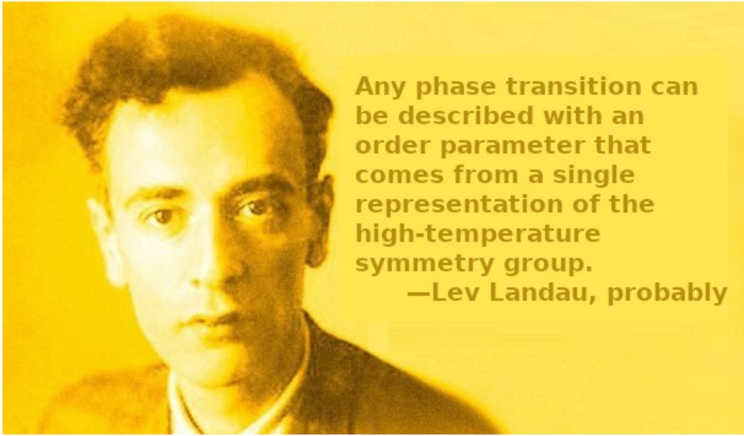

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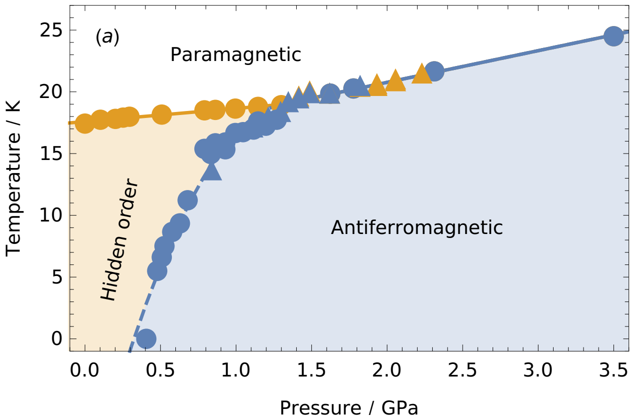
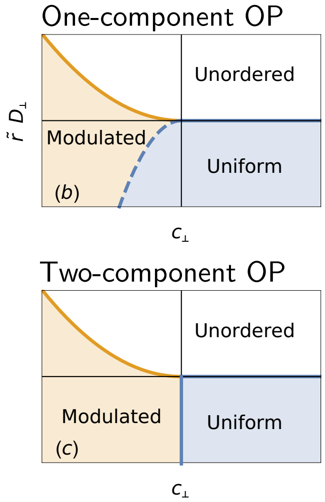
Mean-field phase diagrams





Hidden Order in URu_2Si_2

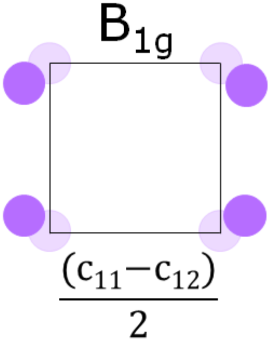
Mean-field modulus

Elastic modulus for strain with symmetry of order parameter is

$$C = C_0 \left(1 + \frac{A}{C_0} \frac{1}{q_*^4 + B|\Delta T|} \right)^{-1}$$

for q_* the modulation wavevector.

Only one representation is consistent with this behavior!



Hidden Order in URu_2Si_2

Comparison with data

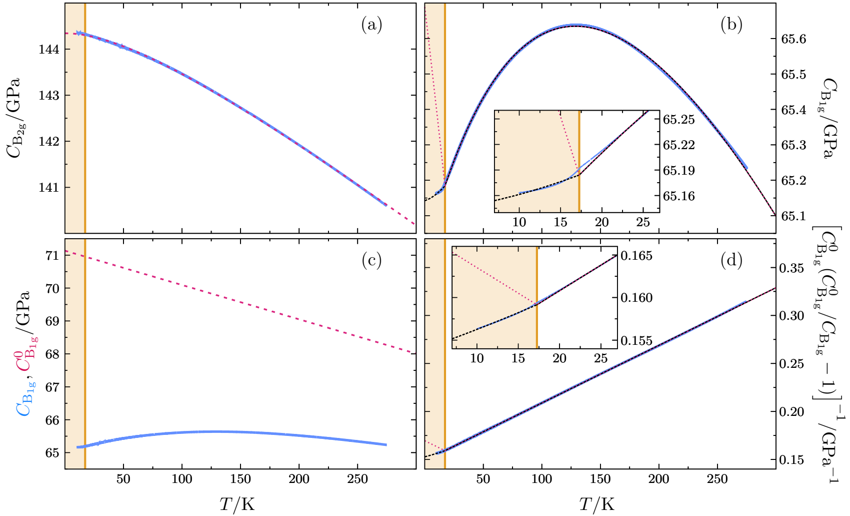


Fig. 1: “Novel,” fracture surface of a fuse network with quenched disorder.

Fig. 2: “Critical,” hard spheres colored by the argument of their hexatic order parameter.

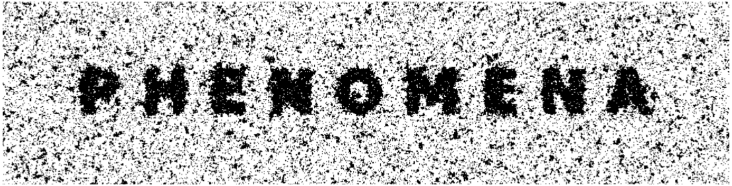


Fig. 3: “Phenomena,” constant-magnetization Ising model in its high-temperature phase.